Solving the steady state diffusion equation with uncertainty

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Abstract

- **Goal:** to efficiently solve a steady state diffusion equation with a random coefficient.
- Monte-Carlo methods are time intensive.
- Using principal components analysis (also known as the Karhunen-Loéve expansion) allows the random coefficient to be approximated with a finite sum of random variables.
- This expansion combined with a stochastic finite element method should reduce computation time.
Problem

The equation to be solved is

$$- \nabla \cdot (c(x, \omega) \nabla u) = f(x),$$  \hspace{1cm} (1)$$

where the diffusion coefficient is a random field.

- $c$ takes the form $c = e^{a(x, \omega)}$ to ensure that it is positive for all $x$. This guarantees existence and uniqueness of the solution of Equation (1).
- Assume a bounded spatial domain $D \subset \mathbb{R}^2$.
- The boundary conditions are deterministic.

$$u(x, \omega) = g(x) \text{ on } \partial D_D$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial D_n.$$
Problem

- The solution is a function of the sample space from which quantities such as the moments or cumulative distribution functions can be found.

- Applications include modeling groundwater flow through a porous medium.
In general, the structure of the diffusion coefficient is unknown.

Previous work has been done where the log of the diffusion coefficient, \( a(x, \omega) \), is written as an infinite series expansion of random variables [1],[4].

The random field can then be approximated by a finite number of terms in this expansion.

This project will instead look at the series expansion of \( c(x, \omega) \).
Approach

- Determine the covariance at each pair of points on the spatial domain.

\[ C(x, y) = \int_{\Omega} (c(x, \omega) - \mu(x))(c(y, \omega) - \mu(y))dP(\omega) \]

\[ C_{ij} = \frac{1}{n} \sum_{k=1}^{n} (c(x_i, \omega_k) - \hat{\mu}_i)(c(x_j, \omega_k) - \hat{\mu}_j) \]

- The mean, \( \mu(x) \) is defined as

\[ \mu(x) = \int_{\Omega} c(x, \omega)dP(\omega) \]

- Under the assumption that the random field is stationary, the mean and variance are constant at each point on the domain.
Approach

- Find the eigenpairs of

\[ C c(x) = \int_D C(x, y)c(y)dy = \lambda c(x) \]

- An expansion for the random field in terms of uncorrelated random variables is given as

\[ c(x, \omega) = \mu(x) + \sum_{s=1}^{\infty} \sqrt{\lambda_s} c_s(x) \xi_s(\omega) \]

- Keeping the first \( M \) terms provides an approximation for the random field.
Weak formulation

Find \( u \in H^1(D) \times L^2(\Omega) \) such that

\[
a(u, v) = l(v), \quad \forall v \in H^1_0(D) \times L^2(\Omega).
\]

\[
a(u, v) = \int_{\Omega} \int_D c(x, \omega) \nabla u(x, \omega) \cdot \nabla v(x, \omega) dx \ dP(\omega)
\]

\[
l(v) = \int_{\Omega} \int_D f(x)v(x, \omega) dx \ dP(\omega)
\]
Stochastic collocation method

- The physical space, $H^1(D)$, and probability space, $L^2(\Omega)$, are discretized separately.
- Because the random field is represented as a finite expansion of random variables, consider $L^2(\Gamma)$.
- A number of points, known as collocation points are selected from $\Gamma$.
- The deterministic finite element method is used to discretize $H^1(D)$ and to find the solution at each collocation point.
- Lagrange interpolation is used to find an approximation of $u$ for points not in the set of collocation points.
The stochastic Galerkin method is similar to stochastic collocation, except the discretization is found for the entire space.

The stochastic discretization comes from polynomials of the random variables, where increasing the degree of the polynomials improves the approximation.

This produces a larger matrix that is to be solved using a Galerkin finite element method.

However, certain aspects of the structure of this matrix and/or its sparsity can be used to reduce this computation time– see [2],[4].
Issues

- How best to discretize the problem in space?
- How do we find a probability density function for $\eta_s(\omega)$ and/or sample it?
- Which approach to use? Galerkin vs. Collocation method
- Will preconditioning be used? – for more about this see [2].
- How many terms to keep in the series? Does this compare to the results when $a(x, \omega)$ was expanded?
Implementation

- Computer: Desktop with 1.9 GB RAM
- Language: Matlab R2008b
- Some previous code may be used for the Galerkin method and preconditioning.
Validation

- One way to solve this problem is using Monte Carlo simulations.
- For each sample of $c(x, \omega)$, the resulting pde can be solved using a deterministic finite element method.
- The moments from the Monte Carlo method will be compared to the results of the stochastic finite element method.
Milestones

Stage 1: October-Late November
- Clearly define the problem (what assumptions will be made?)
- Build the covariance matrix
- Compute the eigennodes
- Write code which generates Monte-Carlo solutions

Stage 2: Late November-December
- Run the Monte-Carlo simulations
- Begin construction of the principal components analysis
Milestones

Stage 3: December - late February
- Complete construction of PCA
- Write solution method

Stage 4: March - April
- Run numerical method
- Analyze accuracy and validity of the method
- Draw conclusions
Deliverables

- Code that calculates the moments of the solution to equation (1) using a Monte-Carlo method
- Code that calculates the moments of the solution to equation (1) using a KL expansion and stochastic evaluation technique
- Comparison of the results for a varying number of terms in the KL expansion
- Comparison of computational cost between the two methods


