Turan Nazarov inequality

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Langer's lemma

Theorem

Let $p(z) = \sum_{k=1}^{n} c_k e^{i\lambda_k z}$ $(0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n = \lambda)$ *be an exponential polynomial that not identically zero. Then the number of complex zeros of* $p(z)$ *in any open vertical strip* $x_0 < Re\ z < x_0 + \Delta$ *of width* Δ *does not exceed* $(n-1) + \frac{\lambda\Delta}{2\pi}$.

Proof

Without loss of generality, we can assume that the coefficients c_1 and c_n are different from zero and that there are no zeros of the exponential polynomial $p(z)$ at the boundary of the strip $x_0 < \text{Re } z < x_0 + \Delta$.

We apply the argument principle for estimating the number of zeros $p(z)$ in the rectangle $Q = \{z : x_0 < \text{Re } z < x_0 + \Delta, |\text{Im } z| \leq y\}$ when $y \to \infty$.

- On the upper side of Q, $p(z) = c_1 + O(e^{-\lambda_2 y})$ and, thus, the increment of the argument along it tends to 0 as $y \to \infty$.
- $p(z) = c_n e^{i\lambda z}(1 + O(e^{-(\lambda - \lambda_{n-1})y}))$ on the lower side of Q. The increment of the argument along it tends to $\lambda\Delta$ as $y \to \infty$.

- The increment of the argument $p(z)$ on any vertical segment $\{z = x + it : t \in [\alpha, \beta]\}$, free from zeros of p, does not exceed $\pi(n-1)$.

Let $\xi := e^{i\arg\, p(x_0+i\alpha)}$,
$q(t) := \operatorname{Im}\, \bar{\xi} p(x_0 + it) = \sum_{k=1}^{n} a_k e^{-\lambda_k t}$ ($a_k = \operatorname{Im}\, \bar{\xi} c_k e^{i\lambda_k x_0}$) - a real exponential polynomial. $q(\alpha) = 0$.

If $q \equiv 0$, then for $t \in [\alpha, \beta]$, $p(x_0 + it)$ lies on the ray $\{\xi y : y > 0\}$. So, $\Delta_{[\alpha,\beta]}\arg\, p(x_0 + it) = 0$.

Otherwise, real zeros of $q(t)$ split the segment $[\alpha, \beta]$ into at most $n-1$ segments I_j. Within each segment I_j, $p(x_0 + it)$ lie in one of the half-planes defined by the straight line $\{\xi y : y \in \mathbb{R}\}$ and thus $|\Delta_{I_j} \arg p(x_0 + it)| \leq \pi$. So, $|\Delta_{[\alpha,\beta]} \arg p(x_0 + it)| \leq (n-1)\pi$.

The increment of the argument on each side of Q is $\leq \pi(n-1)$.

Putting it together, the upper bound for the $|\Delta \arg p|$ when traversing the boundary Q goes to $2\pi \left(\frac{\Delta\lambda}{2\pi} + (n-1)\right)$ as $y \to \infty$

Theorem

Let $p(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t}$*, where* $c_k \in \mathbb{C}$*,* $\lambda_1 < \cdots < \lambda_n \in \mathbb{R}$*,* E *is a measurable subset of an interval* I*. Then*

$$\sup |p(t)|_{t \in I} \leq \left(\frac{316\mu(I)}{\mu(E)} \right)^{n-1} \sup_{t \in E} |p(t)|$$

With a linear change of variables, it is enough to show:

Theorem

Let $p(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t}$*, where* $c_k \in \mathbb{C}$*,* $\lambda_1 < \cdots < \lambda_n \in \mathbb{R}$*,* E *is a measurable subset of the segment* $I = [-\frac{1}{2}, \frac{1}{2}]$*. Then*

$$\sup |p(t)|_{t \in I} \leq \left(\frac{316}{\mu(E)} \right)^{n-1} \sup_{t \in E} |p(t)|$$

Case 1: $\lambda \leq n-1$

Idea:
We want to remove a "bad set" of measure $< \mu(E)$ close to the zeros of p. Zeros are well separated by Langer's lemma.

WLOG, we can assume that $0 = \lambda_1 < \cdots < \lambda_n = \lambda \leq n-1$.
Complex zeros of the z_j exponential polynomials $p(z)$ are well separated by Langer's lemma.
$\#$ zeros in a vertical strip of width $\Delta \leq \frac{\Delta\lambda}{2\pi} + n - 1 \leq \left(1 + \frac{\Delta}{2\pi}\right)(n-1)$

Number z_j in non-decreasing order of $|\mathrm{Re}\ z_j|$. Then for all $j \in \mathbb{N}$

$$|\mathrm{Re}\ z_j| \geq \pi \frac{j-(n-1)}{n-1}$$

(Otherwise, the zeros $z_1, \ldots, z_j$ lie in a vertical strip of width $\Delta < 2\pi \frac{j-(n-1)}{n-1}$ and by Langer's lemma, it has $\left(1 + \frac{\Delta}{2\pi}\right)(n-1) < j$ zeros).

The Hadamard factorization for $p(z)$:

$$p(z) = ce^{az} \prod_{j=1}^{2(n-1)} (z - z_j) \prod_{j>2(n-1)} \left(1 - \frac{z}{z_j}\right) e^{z/z_j} \stackrel{\text{def}}{=} ce^{az} Q(z) R(z)$$

As $j > 2(n-1)$, $|\text{Re } z_j| > \pi$. $R(z)$ is "well behaved" on the strip $\frac{-1}{2} < \text{Re z} < \frac{1}{2}$.

Estimating $R(z)$

As $j > 2(n-1)$, $|\text{Re } z_j| > \pi$. Now we have for $|\text{Re } z| < \frac{1}{2}$

$$\left|\frac{d}{dz}\log R(z)\right| = \left|\sum_{j>2(n-1)}\left(\frac{1}{z_j}+\frac{1}{z-z_j}\right)\right| \leq |z|\left|\sum_{j>2(n-1)}\frac{1}{|z_j||z-z_j|}\right|$$

$$\leq |z|\left|\sum_{j>2(n-1)}\frac{1}{|\text{Re } z_j|(|\text{Re } z_j|-|\text{Re } z|)}\right|$$

$$\leq |z|\left|\sum_{j>2(n-1)}\frac{1}{|\text{Re } z_j|(|\text{Re } z_j|-1/2)}\right|$$

$$\leq 2|z|\sum_{j=1}^{\infty}\frac{1}{|\text{Re } z_{2(n-1)+j}|^2}$$

$$\leq 2|z| \sum_{j=1}^{\infty} \frac{1}{|\text{Re } z_{2(n-1)+j}|^2}$$

$$\leq 2|z| \sum_{j=1}^{\infty} \frac{1}{\left(\pi + \frac{\pi j}{n-1}\right)^2} \leq \frac{2(n-1)|z|}{\pi} \int_{\pi}^{\infty} \frac{dt}{t^2}$$

$$= \frac{2|z|}{\pi^2}(n-1)$$

We have proved

$$\left| \frac{d}{dz} \log R(z) \right| \leq \frac{2|z|}{\pi^2}(n-1)$$

$$\Rightarrow \int_{-1/2}^{1/2} \left| \frac{d}{dz} \log R(z) \right| dz \leq \frac{2(n-1)}{\pi^2} \int_{-1/2}^{1/2} |z| dz = \frac{n-1}{2\pi^2}$$

$$\Rightarrow \max_{z \in [-\frac{1}{2}, \frac{1}{2}]} |R(z)| \leq \exp\left(\frac{n-1}{2\pi^2}\right) \min_{z \in [-\frac{1}{2}, \frac{1}{2}]} |R(z)|$$

Estimating e^{az}

$$\Delta_{[-i\omega,i\omega]}\arg\ p = 2\omega \text{Re}\ a + \Delta_{[-i\omega,i\omega]}\arg\ Q + \Delta_{[-i\omega,i\omega]}\arg\ R$$

- $$|\Delta_{[-i\omega,i\omega]}\arg\ p| \leq \pi(n-1)$$
- $$\Delta_{[-i\omega,i\omega]}\arg\ Q| \leq 2\pi(n-1)$$
- $$|\Delta_{[-i\omega,i\omega]}\arg\ R| \leq \int_{-\omega}^{\omega} \left| \frac{d}{dz} \log R(it) \right| dt \leq \frac{n-1}{\pi^2} \int_{-\omega}^{\omega} |t| dt = \frac{n-1}{\pi^2}\omega^2$$
- $$\Delta_{[-i\omega,i\omega]}\arg(ce^{az}) = 2\omega \text{Re}\ a$$

We conclude that

$$|\text{Re } a| \leq \min_{\omega > 0} \left(\frac{3\pi}{2\omega} + \frac{\omega}{2\pi^2} \right) (n-1) = \sqrt{\frac{3}{\pi}}(n-1)$$

and so,

$$\begin{aligned} \max_{z \in [-\frac{1}{2}, \frac{1}{2}]} \left| \frac{p(z)}{Q(z)} \right| &\leq \exp \left(\left(\sqrt{\frac{3}{\pi}} + \frac{1}{2\pi^2} \right) (n-1) \right) \min_{z \in [-\frac{1}{2}, \frac{1}{2}]} \left| \frac{p(z)}{Q(z)} \right| \\ &\leq 3^{n-1} \min_{z \in [-\frac{1}{2}, \frac{1}{2}]} \left| \frac{p(z)}{Q(z)} \right| \end{aligned}$$

Estimating $Q(z)$

Let $0 < h < 1/8$.

Let n_1 be the largest natural number for which there exists a circle D_1 of radius $\frac{n_1}{n-1}h$ containing at least n_1 zeros of the polynomial Q.

D_1 contains exactly n_1 zeros of Q, otherwise it can be increased.

Let n_2 be the largest number for which there exists a circle D_2 of radius $\frac{n_2}{n-1}h$ containing at least n_2 zeros Q from among those not included in D_1 etc. until we cover all the zeros.

Let us put $D'_k = 2D_k$ (a circle with the same center, but twice the radius).

We get a sequence of numbers $n_1 \geq \cdots \geq n_s$ with

$$n_1 + \cdots + n_s = 2(n-1)$$

and the corresponding circles $D'_1, \ldots, D'_s$ with a sum of radii of $4h$.

Let $z \in [-\frac{1}{2}, \frac{1}{2}] \setminus \cup_{i=1}^{s} D'_i$. Let us renumber the zeros of Q in the order of non-decreasing $|z - z_j|$.

$$|z - z_j| \geq \frac{j}{n-1} h$$

Otherwise, the circle D with center z and radius $\frac{j}{n-1} h$ contains at least j zeros of Q. Let us choose $m \in \{1, ..., s\}$ so that $n_1 \geq \cdots \geq n_m \geq j > n_{m+1} \geq \cdots \geq n_s$.

Because $z \notin \cup_{k=1}^{s} D'_k$, for $l \leq m$,

$$\text{dist}(z, \text{center}(D_k)) \geq \frac{2n_k}{n-1} h \geq \frac{n_k}{n-1} h + \frac{j}{n-1} h$$

D does not intersect with any of the disks $D_1, \ldots, D_m$ and has j zeros.

But then at step $m+1$ we should have taken not D_{m+1}, but D (or a circle with an even larger number of zeros), a contradiction.

Langer's lemma gives the inequality

$$|z - z_j| \geq \pi \frac{j - (n-1)}{n-1}$$

Otherwise the zeros $z_1, \ldots, z_j$ would lie in a circle of radius less $\pi \frac{j-(n-1)}{n-1}$.
And those moreover, in a band of width $\Delta < 2\pi \frac{j-(n-1)}{n-1}$).
So, we have

for $z \in [-\frac{1}{2}, \frac{1}{2}] \setminus \cup_{i=1}^{s} D_i'$

$$|z - z_j| \geq \pi \frac{j - (n-1)}{n-1}$$

$$|z - z_j| \geq \frac{j}{n-1} h$$

$$
\begin{aligned}
\frac{|Q(z)|}{\max\{|Q(t) : t \in [-\frac{1}{2}, \frac{1}{2}]\}} &\geq \prod_{j=1}^{2(n-1)} \frac{|z - z_j|}{\max\{|t - z_j| : t \in [-\frac{1}{2}, \frac{1}{2}]\}} \\
&\geq \prod_{j=1}^{2(n-1)} \frac{|z - z_j|}{1 + |z - z_j|} \\
&= \prod_{j=1}^{n-1} \frac{|z - z_j|}{1 + |z - z_j|} \prod_{j=1}^{n-1} \frac{|z - z_{n-1+j}|}{1 + |z - z_{n-1+j}|} \\
&\geq \prod_{j=1}^{n-1} \frac{\frac{j}{n-1} h}{1 + \frac{j}{n-1} h} \prod_{j=1}^{n-1} \frac{\frac{\pi j}{n-1}}{1 + \frac{\pi j}{n-1}} \\
&> (8h)^{n-1} \times \prod_{j=1}^{n-1} \frac{\frac{j}{n-1} \frac{1}{8}}{1 + \frac{j}{n-1} \frac{1}{8}} \prod_{j=1}^{n-1} \frac{\frac{3j}{n-1}}{1 + \frac{3j}{n-1}}
\end{aligned}
$$

But for any $\theta > 0$,

$$\prod_{j=1}^{n-1} \frac{\frac{\theta j}{n-1}}{1+\frac{\theta j}{n-1}} \geq \exp\left((n-1)\int_0^1 \log\left(\frac{\theta t}{1+\theta t}\right) dt\right) = \left(\frac{\theta}{(1+\theta)^{1+1/\theta}}\right)^{n-1}$$

So,

$$\frac{|Q(z)|}{\max\{|Q(t) : t \in [-\frac{1}{2}, \frac{1}{2}]\}} > (8h)^{n-1}\left(8 \times \left(\frac{9}{8}\right)^9 \times \frac{4\sqrt[3]{4}}{3}\right)^{-(n-1)}$$
$$\geq \left(\frac{8h}{32\sqrt[3]{4}}\right)^{n-1}$$

The measure of the exceptional set

$$\mu\left([-\frac{1}{2},\frac{1}{2}]\cap(\cup_{k=1}^{s}D'_k)\right) < 8h$$

If $h < \frac{\mu(E)}{8}$, then

$$[-\frac{1}{2},\frac{1}{2}]\cap(\cup_{k=1}^{s}D'_k)\cap E \neq \phi$$

Combining estimates so far,

$$\sup|p(t)|_{t\in I} \leq \left(\frac{96\sqrt[3]{4}}{\mu(E)}\right)^{n-1}\sup_{t\in E}|p(t)| \leq \left(\frac{154}{\mu(E)}\right)^{n-1}\sup_{t\in E}|p(t)|$$