1. (4 points) Find $\frac{dy}{dx}$
   a) $y = \sin(\sin(x))$
   b) $y = \frac{1}{\ln x}$
   c) $y = \sqrt{x^4 + 2x^2 - 1}$
   d) $y = x^x$

   **Answer:**
   a) $\frac{dy}{dx} = \cos(\sin(x)) \cdot \frac{d}{dx} \sin(x) = \cos(\sin(x)) \cos(x)$
   b) $\frac{dy}{dx} = -\frac{1}{(\ln x)^2} \cdot \frac{d}{dx} \ln x = -\frac{1}{(\ln x)^2} \cdot \frac{1}{x}$
   c) $\frac{dy}{dx} = \frac{1}{2} (x^4 + 2x^2 - 1)^{-1/2} \cdot \frac{d}{dx} [x^4 + 2x^2 - 1] = \frac{1}{2} (x^4 + 2x^2 - 1)^{-1/2} (4x^3 + 4x)$
   d) Since $b^a$ means $e^{a \ln b}$, so $x^x = e^{\ln x^x}$, then $\frac{dy}{dx} = x^x \ln x \cdot \frac{d}{dx} [x \ln x] = x^x (1 \cdot \ln x + x \cdot \frac{1}{x})$. This simplifies to $x^x (\ln x + 1)$.

2. (4 points) Draw the tangent line to a point $(x_0, y_0)$ on the hyperbola defined by $x^2 - y^2 = 1$. Find an equation for the $x$-intercept of this line in terms of $x_0$.

   **Answer:** Implicitly differentiate with respect to $x$
   $$2x - 2y \frac{dy}{dx} = 0$$
   therefore, $\frac{dy}{dx} = x/y$. Thus the tangent line at the point $(x_0, y_0)$ has slope $x_0/y_0$. The equation of the tangent line in point-slope form is
   $$y - y_0 = \frac{x_0}{y_0} (x - x_0)$$
   The $x$-intercept of this line is where $y = 0$. In other words, $-y_0 = \frac{x_0}{y_0} (x - x_0)$, solving for $x$ gives $x = \frac{-y_0^2 + x_0^2}{x_0} = \frac{1}{x_0}$, since $x_0^2 - y_0^2 = 1$.

3. (4 points) A naval destroyer is tracking a submarine 300 meters below the ocean. Using sonar, it is determined that the distance between the submarine and the destroyer is exactly 1 kilometer, and increasing at 75 kilometers per hour. How fast is the submarine traveling?

   **Answer:** The Pythagorean theorem relates the distance, $d$, between the destroy and the submarine with the horizontal distance, $x$, the the submarine has traveled:
   $$d^2 = 300^2 + x^2.$$
Differentiate with respect to $t$ to get $2d\frac{dd}{dt} = 2x\frac{dx}{dt}$. Under the initial conditions given, $d = 1000$ and $\frac{dd}{dt} = 75000$. We determine $x = 100\sqrt{91} \approx 954$ using the above equation. Thus

$$\frac{dx}{dt} \approx \frac{2 \cdot 1000 \cdot 75000}{2 \cdot 954} \approx 78616$$

4. (4 points) Estimate $\sin^2(4\pi/9)$.

**Answer:** Let $f(x) = \sin^2(x)$. $\frac{3\pi}{9} \approx \frac{4\pi}{9}$. So let $a = \frac{3\pi}{9} = \frac{\pi}{3}$, $x = \frac{4\pi}{9}$, so $h = \frac{\pi}{9}$. Also, $f'(x) = 2\sin(x)\cos(x)$. By linear approximation,

$$f(x) \approx f(a) + f'(a) \cdot h = \sin^2 \left( \frac{\pi}{3} \right) + 2 \sin \left( \frac{\pi}{3} \right) \cos \left( \frac{\pi}{3} \right) \cdot \frac{\pi}{9}$$

$$= \left( \frac{\sqrt{3}}{2} \right)^2 + \frac{2\sqrt{3}}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{9} = \frac{3}{4} + \frac{\sqrt{3}\pi}{18}$$

5. (4 points) The golden ratio is the positive number $\phi$ satisfying $\phi^2 - \phi - 1 = 0$. Approximate the golden ratio using two iterations of Newton’s method starting at $c_0 = 1$.

**Answer:** Here $f(x) = x^2 - x - 1$. Then $f'(x) = 2x - 1$, so

$$c_1 = c_0 - \frac{f(c_0)}{f'(c_0)} = 1 - \frac{1 - 1 - 1}{2 - 1} = 1 - (-1) = 2$$

Again,

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 2 - \frac{4 - 2 - 1}{4 - 1} = 2 - \frac{1}{3} = \frac{5}{3}$$

Newton’s method estimates a solution to $x^2 - x - 1 = 0$ to be about $5/3$.

6. (4 points) Implicitly differentiate the equation

$$\ln(xy) = \frac{x}{y}$$

twice and solve for $\frac{d^2y}{dx^2}$. Your answer should have no $\frac{dy}{dx}$ in it.

**Answer:** Implicitly differentiate

$$\frac{1}{xy} \cdot \frac{d}{dx}(xy) = \frac{y \cdot 1 - x \cdot \frac{dy}{dx}}{y^2}$$

$$\frac{1}{xy} \cdot (1 + x \cdot \frac{dy}{dx}) = \frac{y \cdot 1 - x \cdot \frac{dy}{dx}}{y^2}$$

$$\frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{y} - \frac{x}{y^2} \cdot \frac{dy}{dx}$$
\[
\left(\frac{1}{y} + \frac{x}{y^2}\right) \frac{dy}{dx} = \frac{1}{y} - \frac{1}{x} \\
\frac{dy}{dx} = \frac{1}{y} - \frac{1}{x} = \frac{xy - y^2}{xy + x^2}
\]

Now differentiate again

\[
\frac{d^2 y}{dx^2} = \frac{(xy + x^2)(y + x \frac{\frac{dy}{dx}}{x^2} - 2y \frac{dy}{dx}) - (xy - y^2)(y + x \frac{\frac{dy}{dx}}{x^2} + 2x)}{(xy + x^2)^2}
\]

\[
\frac{d^2 y}{dx^2} = \frac{(xy + x^2)(y + x \frac{\frac{1}{y} - \frac{1}{x} - \frac{1}{y} + \frac{1}{x}}{y + x^2} - 2y \frac{\frac{1}{y} - \frac{1}{x} + \frac{1}{y} + \frac{1}{x}}{y + x^2}) - (xy - y^2)(y + x \frac{\frac{1}{y} - \frac{1}{x} + \frac{1}{y} + \frac{1}{x}}{y + x^2} + 2x)}{(xy + x^2)^2}
\]

Oops that problem sucked