THE GEOMETRY OF THE ADELES

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Abstract. Abstract

Contents

1. Introduction

Let K be a local field. According to local class field theory, an arithmetic object ($K^\times$) contains all the information to understand the maximal abelian extensions of K. This correspondence is contained in the local Artin map, for any unramified extension L/K,

$$K^\times/NL^\times \to \text{Gal}(L/K)$$

[1] What is the corresponding object for number fields?

2. Ideles

2.0.1. *Valuations.* A valuation on a field is a group homomorphism $v : K^\times \to \mathbb{R}$ satisfying

$$v(x+y) \geq \min\{v(x), v(y)\}$$

for all $x, y \in K$, where in passing we mention the necessary convention $v(0) = \infty$. A valuation is defined to be normalized if its image is $\mathbb{Z}$,[2] and a valuation combined with a real number $0 < \alpha < 1$ defines a metric on K,

$$|x|_v = \alpha^{v(x)}.$$

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[1]correct: and taking limits,

$$K^\times \to \text{Gal}(K^{un}/K)$$

$\widehat{K^\times}$?

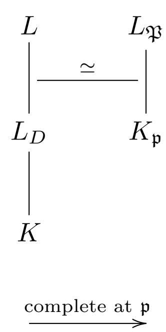
[2]false. we only call it that if it is a non-Archimedean valuation. If it's Archimedean then we require it to be $||$ or $||^2$ dep. on real/complex...

2.0.2. *Example.* Let $\mathcal{O}$ be a Dedekind domain, K its field of fractions. Then every prime ideal $\mathfrak{p} \subset \mathcal{O}$ defines a valuation on K in the following manner. For any $x \in K^\times$, write the fractional ideal $x\mathcal{O} \subset K$ as a product of prime ideals

$$x\mathcal{O} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$$

We define $\mathrm{ord}_\mathfrak{p}(x)$ to be the exponent of $\mathfrak{p}$ appearing in the above product, $\mathrm{ord}_\mathfrak{p}(x) = 0$ if $\mathfrak{p}$ does not appear. Since $\mathcal{O}$ is a Dedekind domain, the factorization exists and is unique, so $\mathrm{ord}_\mathfrak{p}$ is well defined. It is easy to check that it is a valuation.

If K is a number field, then K will not be complete with respect to this valuation. Therefore we form its completion, $K_\mathfrak{p}$ and consider K as a subfield. The philosophy ("local-global/Hasse principle") is that the collection of *all* $K_\mathfrak{p}$, for *all* $\mathfrak{p}$ contain any data you might want about K. The first fact one proves is that if L/K is a Galois extension and $\mathfrak{P}$ is an ideal of L over $\mathfrak{p}$ an ideal of K, then the decomposition group of $\mathrm{Gal}(L/K)$ is canonically isomorphic to $\mathrm{Gal}(L_\mathfrak{P}/K_\mathfrak{p})$.



Therefore we should try and forget about the ideal $\mathfrak{p}$, and instead consider all of the metrics K obtains from sitting inside its completions. Because of this philosophy, we refer to the valuations $v_\mathfrak{p}$ as places, and just write them as v without the $\mathfrak{p}$.

2.0.3. An **absolute value** is a (nontrivial) homomorphism $|\cdot| : K^\times \to \mathbb{R}_{\geq 0}$ which we extend as $|0| = 0$ satisfying,

$$|x + y| \leq |x| + |y|$$

up to equivalence, where two absolute values are equivalent if they induce the same metric on K (equivalently, if $|\cdot|_1 = |\cdot|_2^c$ for $c > 0$).

We have the important theorem:

Theorem 1 (Ostrowski)**.** *The absolute values on $\mathbb{Q}$ are exactly those induced by its prime ideals (p) and the euclidean metric $|\cdot|$.*

We can make an argument about extending valuations and get the following,

Corollary 2. *Let K be a number field with real embeddings $\sigma_1, \ldots, \sigma_{r_1}$ and complex embeddings $\tau_1, \overline{\tau}_1, \ldots, \tau_{r_2}, \overline{\tau}_{r_2}$, so $r_1 + 2r_2 = [K : \mathbb{Q}]$.*

The absolute values on a number field K are exactly those induced by its prime ideals $\mathfrak{p}$, and the $r_1 + r_2$ Archemedian absolute values $|x|_i = |\sigma_i(x)|$ and $|x|_j = |\tau_j(x)|^2$.

2.0.4. *The ideles.* Throughout, K is a number field and S is a finite set of places including all Archimedian places, which we always denote S_∞. However, a similar version of all statements is true for global function fields.

We define the **S-ideles** to be the group

$$I_S = \prod_{v \in S} K_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$$

that is, vectors where the coordinates lie in all the completions of K, and are v-adic units for $v \notin S$. If we give the S-ideles the product topology then they are a abelian topological group. Since each $\mathcal{O}_v^\times$ is compact and the $K_v^\times$ are locally compact, it follows that the S-ideles are locally compact.[3]

Define the **ideles** to be the group

$$I = \bigcup_S I_S \subset \prod_v K_v^\times.$$

There are many ways to understand the topology on the ideles. For the moment however, we'll just say there is a unique topology on I such that the subspace topology on I_S is the same one defined above. If a typical element of I is x, its coordinate in the v place will be denoted x_v.

Historically, the ideles were thought of as a generalization of ideals. Via the map,

$$I \xrightarrow{id} \{\text{ideal class group}\}$$
$$(a_v) \overset{\mathfrak{a}}{\longmapsto}$$
$$\text{where } \mathfrak{a} = \{x \in K \mid |x|_v \leq |a_v|_v, \text{ all finite } v\}$$

Exercise Show that if $\alpha \in K^\times$ then $id(\alpha) = [\alpha\mathcal{O}]$, and if $a \in I_{S_\infty}$ then $id(a) = [\mathcal{O}]$. That is, $K^\times I_{S_\infty} \subset \ker(id)$ (It helps to find another description of the map id using unique factorization of ideals).

Conversely if $x = (x_v) \in \ker(id)$ then it represents a principal ideal, thus there exists $\alpha \in K^\times$ such that $|x_v|_v = |\alpha|_v$ for all finite v. Thus $x\alpha^{-1} \in I_{S_\infty}$ and we've shown an isomorphism

$$id : I/K^\times I_{S_\infty} \xrightarrow{\sim} \{\text{ideal class group}\}$$

2.0.5. *Idele norm.* The ideles have a metric $I \to \mathbb{R}_{>0}^\times$ defined for $x = (x_v)$ as

$$|x| = \prod_v |x_v|_v$$

[3]Remark: It is *not* a consequence of Tychonoff that the infinite product of locally compact spaces is locally compact. However, we are forcing all but finitely many of the spaces to be compact.

since $x \in I_S$ for some S, x is a v-adic unit for all but finitely many v, so this is a well defined (finite) product.

Fact: this metric defines the same topology on the ideles as above.

Define the "unit circle",

$$I^1 = \{x \in I \mid \prod_v |x_v|_v = 1\} = \ker(x \mapsto |x|)$$

The following theorem, whose proof is sketched later, is essential.

Theorem 1. *$K^\times$ is a discrete and cocompact subset of I^1.*

Proof. Discreteness will follows from the fact that K is discrete in A and that I^1 has a closed embedding into A^2. We won't show cocompactness ■

We define the compact quotient $C^1 = I^1/K^\times$ to be the **norm-1 idele class group** .

Corollary 2 (Finiteness of the class number)**.**

$$\#\{\textit{ideal class group}\} < \infty.$$

Proof. Claim: $I/K^\times I_{S_\infty}$ is isomorphic to $I^1/K^\times I^1_{S_\infty}$. (Proof omitted)

I_{S_∞} is open in I, so $I^1_{S_\infty}$ is open in I^1. Let H denote its image in the quotient C^1. Quotient maps are open, so H is an open subgroup. Its cosets form a disjoint open cover of the compact group C^1, thus there can only be finitely many cosets. By the claim, $I/K^\times I_{S_\infty}$ is isomorphic to the quotient C^1/H, and we previously showed the former was isomorphic to the ideal class group of K. ■

Remark: In general we can define the S-ideal class group and a similar proof shows it is finite.

Corollary 3 (Dirichlet's unit theorem)**.** *Let $r = \#(S_\infty) - 1$. Then*

$$\mathcal{O}^\times \simeq \mu \times \mathbb{Z}^r$$

where μ denotes the roots of unity of K.

Proof. Consider the "adelic torus"

$$T = \{x \in I \mid |x|_v = 1 \text{ for all } v\}.$$

Then T is compact (Tychonoff), and $T \subset I_{S_\infty}$. Similarly define the "adelic quadrant"

$$U = \{x \in I_{S_\infty} \mid x_v > 0 \text{ for all } v \mid \infty\},$$

where if v is complex, $x_v > 0$ means $x_v \in \mathbb{R}^\times_{>0} \subset \mathbb{C}^\times = K_v^\times$.

Exercise: We have a split exact sequence of topological groups,

$$1 \to T \to I_{S_\infty} \to U \to 1.$$

If we restrict the above short exact sequence to $I^1_{S_\infty}$, its image will be the "adelic hyperbola"

$$H = \{x \in I_{S_\infty} \mid \prod_{v \mid \infty} x_v = 1\}$$

and the kernel will still be T,

$$1 \to T \to I^1_{S_\infty} \to H \to 1.$$

The rest of the proof goes as follows:

(1) Not only is H a group, it is actually an $\mathbb{R}$-vector space (written multiplicatively with $\mathbb{R}$ acting in the exponent) of dimension $r-1$.
(2) $T \cap \mathcal{O}_S^\times = \mu$, this is classically proved by looking at the coefficients of minimal polynomials; however here it follows easily from the fact that (compact $\cap$ discrete) = finite.
(3) The image of $I^1_{S_\infty} \cap K^\times = \mathcal{O}_S^\times$ is discrete (this is where discreteness of $K^\times$ is used)
(4) The continuous image of cocompact is cocompact, so the image of $\mathcal{O}_S^\times$ is cocompact in H (this is where the cocompactness of $K^\times$ in $I^1_{S_\infty}$ is used)
(5) A discrete cocompact subgroup of a finite dimensional $\mathbb{R}$-vector space must be a lattice of maximal rank. (This is a standard fact, also proved in the classical case) .[4]

■

Remark: Virtually the same proof holds for *any* $S \subset S_\infty$ (well not quite...), so we should think of Dirichlet's theorem as special case of the more general "S-unit theorem:"

$$\mathcal{O}_S^\times \simeq \mu \times \mathbb{Z}^{\#(S)-1}$$

where we have to replace the exact sequence above with

$$1 \to T \to I^1_S \to H_S \to 1$$

where now

$$H_S = \{x \in I_S \mid \prod_{v \mid \infty} x_v = 1 \text{ and}$$

[4]Remark: I'm a little hesitant about this proof. In all other treatments, they consider the map

$$\mathcal{L} : I_{S_\infty} \to \mathbb{R}^r$$
$$(x_v) \mapsto (\log |x_{\infty_1}|_{\infty_1}, \ldots, \log |x_{\infty_r}|_{\infty_r})$$

for $\infty_i \in S_\infty$. The proof is the same otherwise?

I think it is, you just replace step (1) with,

The image of $I^1_{S_\infty}$ is the hyperplane

$$x_1 + \ldots + x_r = 0$$

which is an $r-1$ dimensional $\mathbb{R}$ vector space.

2.0.6. *The Regulator.* (I don't actually understand this)

We can ask for a fundamental domain of the idele class group, that is, a set $\Omega \subset I$ such that $K^\times \cap \Omega = \{1\}$ and $K^\times \Omega = I$. We can construct a fundamental domain as follows. By the previous section, we can select a basis for the free part of the units, $\mathcal{O}^\times$, say $\{\varepsilon_1, \ldots, \varepsilon_r\}$. Consider the map,

$$\begin{aligned} \mathcal{L} : I_{S_\infty} &\to \mathbb{R}^r \\ (x_v) &\mapsto (\log |x_{\infty_1}|_{\infty_1}, \ldots, \log |x_{\infty_r}|_{\infty_r}) \end{aligned}$$

Let $P \subset \mathbb{R}^r$ be the (a) fundamental parallelotope for the lattice which is the image of our chosen units under $\mathcal{L}$.

Theorem 1 (Regulator)**.**

3. Adeles

The adeles are the "additive" version of the ideles. We're presenting them second because they came later historically, but also because their importance to class field theory is overshadowed by that of the idele class group.

The **S-adeles** are the product,

$$A_S = \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v$$

that is vectors whose coordinates are in all the completions of K, and are v-adic integers for almost all v. The **adeles** are the union,

$$A = \bigcup_S A_S \subset \prod_v K_v$$

Then the adeles form a ring under pointwise addition and multiplication.[5] The units are the ideles,

$$A^\times = I.$$

3.0.7. *Topology.* As with the ideles, we give the adeles the topology called the restricted topology, where sets are declared open if they are open for each coordinate, as in the weak topology, but we also require that they are only $\mathcal{O}_v$ for all but finitely many v.

Equivalently, the adeles get the metric topology from the norm $|\cdot|$ defined in the same way as it was defined for the ideles.[6]

A while ago we mentioned that there are many ways to understand the topology of the ideles $I \subset \mathbb{A}$. What you might expect, that they inherit the subspace topology, is *not* true. Instead we have the following,

[5]As a minor nuisance, the S-adeles do not form a ring unless $S \supset S_\infty$, since the disks $[-1, 1] \subset \mathbb{R}$, $|z| \leq 1 \subset \mathbb{C}$ are not closed under addition, unlike in the non-Archimedean case (this is exactly the non-Archimedean property!)

[6]verify?

Proposition 1. *The following all induce the same (correct) topology on the ideles:*

(1) *The "restricted topology" where the open sets are,*
$$U = \prod_{v \in V} U_v \times \prod_{v \notin V} \mathcal{O}_v^\times$$
where U_v is open in $K_v^\times$ and V is a finite *set of places.*
(2) *The topology is the one which when restricted to I_S gives the product topology.*
(3) *? I don't know if the metric induced by $|.|$ is the topology. I doubt it, since $|.|$ induces (right?) the metric on $\mathbb{A}$ and thus would induced the subspace metric on I.*
(4) *I maps to $\mathbb{A} \times \mathbb{A}$ by,*
$$x \mapsto (x, x^{-1})$$
and the topology is the one which makes this a homeomorphism onto its image.

3.0.8. *Principal adeles.* K is embedded in A diagonally:
$$K \to A$$
$$\alpha \mapsto (\dots, \alpha, \dots)$$

This embedding is easily (pf?) seen to be discrete. Better than in the idele case, we have the following

Theorem 1 (Weak approximation theorem)**.** *$K \subset A$ is cocompact.*

Proof. This is called the weak approximation theorem for the following reason. Define the "fundamental domain" of A/K to be,
$$\Omega = \{x \in A \mid$$

■

3.0.9. *Haar measure.*

Theorem 1. *Let G be a locally compact group. There is a unique, up to a scalar, Radon measure μ such that for all Borel U, $x \in G$,*
$$\mu(xU) = \mu(U) \text{(Left translation invariance)}$$
Such a measure is called a ***Haar measure***

Let φ be any (topological) automorphism of G. Then φ takes Borel sets to Borel sets, and we can define the measure $\mu \circ \varphi$, which will also be a Haar measure. By the uniqueness statement of the above theorem, there exists a number $m_\varphi > 0$ such that,
$$(\mu \circ \varphi)(E) = m_\varphi \mu(E) \qquad \text{for all Borel } E$$

This number is called the **modulus** of the automormophism.

Now consider the Haar measure μ on A. Multiplication by elements of $A^\times = I$ are automorphisms, so what are their moduli?

Proposition 2. *For all $x \in I$,*

$$\mu(xE) = |x|\mu(E) \qquad \textit{for all Borel } E$$

That is, the adele norm $|x| = \prod_v |x_v|_v$ is naturally a geometric quantity: it measures how much a number scales volume. **Proof.** Omitted ■

Example: Let K_v be a local field, v discrete. Then K_v is locally compact, and we choose the one such that,

$$\int_{\mathcal{O}_v} d\mu_v = 1$$

Then multiplication by a uniformizer π divides K_v into a disjoint union of exactly $f = f(\pi \mid p)$ cosets:

$$K_v = \bigcup_{i=1}^{p^f} (i + \pi\mathcal{O}_v)$$

But multiplication by π is an automorphism of K_v, thus these cosets all have the same volume. So by standard properties of measure we must have,

$$\int_{\pi\mathcal{O}_v} d\mu_v = \frac{1}{p^f}$$

which exactly coincides with $|\pi|_v$! That is in general the modulus of x is its v-adic absolute value.

3.0.10. *Artin product formula.* One application of this is the Artin product formula which Tate presents in his thesis. The classical proof is much simpler, but this gives a geometric interpretation.

Proposition 1 (Artin product formula)**.** *For all $x \in K^\times$,*

$$\prod_v |x|_v = 1$$

where the product runs over all places of K.

4. Further Topics

4.0.11. *Class field theory.* Let L/K be a finite extension. From algebraic number theory we know that, if v is a place of K,

$$L \otimes K_v \simeq \prod_{w \mid v} L_w.$$

The corresponding theorem in terms of adeles is that,

$$L \otimes A_K \simeq A_L$$

where this is a *topological* homomorphism. Using this, we can make sense of Galois acting on A_L: it acts scalars L and fixes A_K. Similarly we get a norm map,

$$N_{L/K} : A_L \to A_K$$

There is a continuous surjective homomorphism $\varphi_K : C_K \to \mathrm{Gal}(K^{ab}/K)$ called the **global Artin map** . It is the product of all the local Artin maps (cf. Karpuk's talk):

$$\varphi(a) = \prod_v \varphi_v(a_v)$$

where again recall that $\mathrm{Gal}(L_w/K_v)$ is a subgroup of $\mathrm{Gal}(L/K)$.

Further, the kernel of φ is connected component of the identity in C_K. The kernel is itself a topic of study.

FURTHER READING: J. S. Milne, Class Field Theory

4.0.12. *Regulator.* Can we make sense of the p-adic regulator?

4.0.13. *Tamagawa number.* Let G be a (connected linear) algebraic group defined over K. Then the adeles are a K-algebra, so it makes sense to take the adelic points $G(A)$. Since G is linear, it sits inside affine A space, so inherits a topology which makes it into a locally compact group. Therefore it has a Haar measure, called the **Tamagawa measure** μ. We can consider the subgroup $G(K) \subset G(A)$, and μ will descend to a measure on the homogeneous space $G(A)/G(K)$. The **Tamagawa number** is defined to be $\mu(G(A)/G(K))$.

Example G_m is the algebraic group corresponding to,

$$G_m : XY - 1 = 0 \subset \mathbb{A}^2$$

Then $G_m(A) = I$.

FURTHER READING: Cassels-Froelich, Algebraic Number Theory, Ch. X.

The following is a glossary of the different notations used in different books

TABLE 1. Notations

	CF67	Tat50	MilCFT	RV	NeuCFT	WeiBNT[7]	Others
Adeles (A_K)	V_k	V	$\mathbb{A}_K$	$\mathbf{A}_K$	$\mathbb{A}_K$	$k_{\mathbf{A}}$	$\mathbb{A}_K$
Ideles (I_K)	J_k	I	$\mathbb{I}_K$	$\mathbf{I}_K$	I_K	$k_{\mathbf{A}}^\times$	$\mathbb{A}_K^\times$
S-Ideles ($I_{K,S}$)	J_S	I_S	$\mathbb{I}_S$	$\mathbf{I}_{K,S}$	I_K^S	$k(P)^\times$	$\mathbb{A}_{K,S}^\times$
Norm-1 ideles (I_K^1)	J_k^1	J	$\mathbb{I}^1$	$\mathbf{I}_K^1$	I_K^0	$k_{\mathbf{A}}^1$	?

5. REFERENCES

[CF67] Cassels and Froelich, Algebraic Number Theory. Out of print.(?)

[RV] Ramakrishnan and Valenza, Fourier Analysis on Number Fields. Springer, 1999.(?)

[Tat50] J. T. Tate, Fourier Analysis on Number Fields and Hecke's Zeta-Functions. In Cassels and Froelich.