Solutions to UMD Analysis Quals

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Contents

CHAPTER 1

Solutions to Quals

1. Solutions for August 1999

3. Solution. [1]

(**a**) No.[2] Suppose such a cover exists. Since $\sum m < \infty$, its tails go to 0. So for all $\varepsilon > 0$, there is an N such that

$$\sum_{n=N}^{\infty} m(I_n) < \varepsilon.$$

Since each x is in infinitely many I_n, $\{I_n\}_{n=N}^{\infty}$ still covers E. By monotonicity,

$$m(E) \leq m\left(\bigcup_{n=N}^{\infty} I_n\right) \leq \sum_{n=N}^{\infty} m(I_n) < \varepsilon.$$

Since ε is arbitrary, $m(E) = 0$, contradicting the fact that E has positive measure. $\Rightarrow\Leftarrow$

(**b**) Yes. Since E has measure 0, for every n there is a countable collection of disjoint intervals $\{I_{n,k}\}_{k=1}^{\infty}$ such that $E \subseteq \bigcup_{k=1}^{\infty} I_{n,k}$ and

$$m\left(\bigcup_{k=1}^{\infty} I_{n,k}\right) = \sum_{k=1}^{\infty} m(I_{n,k}) < \frac{1}{2^n}.$$

Then the countable collection of intervals $\{I_{n,k}\}_{n,k=1}^{\infty}$ satisfies the condition that every $x \in E$ is contained in finitely many intervals, and by subadditivity,

$$\sum_{n,k=1}^{\infty} m(I_{n,k}) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} m(I_{n,k}) < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

4. Solution. (**a**) We have the fact that if $t = \overline{\lim}_n |a_n|^{1/n}$, then the power series

Since f, g are analytic on the disk, their radii of convergence are ≥ 1. So

$$\overline{\lim_n}\, |a_n|^{1/n} = t \leq 1 \qquad \text{and,} \qquad \overline{\lim_n}\, |b_n|^{1/n} = s \leq 1.$$

The H-product $f * g$ has coefficients $a_n b_n$ satisfying

$$\overline{\lim_n}\, |a_n b_n|^{1/n} = (\overline{\lim_n}\, |a_n|^{1/n})(\overline{\lim_n}\, |b_n|^{1/n}) = ts \leq 1$$

by basic properties of $\overline{\lim}$. So $f * g$ has radius of convergence ≥ 1, so is analytic on $|z| \leq 1$.

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[1] Remark: this problem is really just the Borel Cantelli lemma: if $\sum m(I_n) < \infty$, the set of $x \in \mathbb{R}$ contained in infinitely many I_n has measure zero. Since that would be a one sentence proof, you should prove it directly.

[2] pf from Xuwen

(**b**) By uniqueness of power series, since $(1-z)^{-1}$ has expansion

$$(1-z)^{-1} = \sum z^n$$

it follows that $|a_n b_n| = 1$. By assumption, both $g \in A$, so $\overline{\lim}_n |b_n|^{1/n} = s \le 1$. So for all but finitely many n, $|b_n|^{1/n} \le 1 \Rightarrow |a_n|^{1/n} \ge 1$. Thus $\overline{\lim}_n |a_n|^{1/n} = t \ge 1$, so the radius of convergence of f is exactly 1.

If f had analytic continuation to every point of $|z| = 1$, it would be analytic on some neighborhood of every such z, and therefore on an neighborhood of the unit circle. Thus its radius of convergence would be strictly greater than 1. $\Rightarrow\!\Leftarrow$[3]

7. Solution. Since $L^2[0,1]$ is a Hilbert space, the parallelogram law holds. Namely for all f_n,

$$\|f_n - f_m\|_2^2 = 2\|f_n\|_2^2 + 2\|f_n\|_2^2 - \|f_n + f_m\|_2^2$$

Taking $\overline{\lim}$ of both sides,[4]

$$\overline{\lim_{n,m}} \|f_n - f_m\|_2^2 = 4\,\overline{\lim_{n}} \|f_n\|_2^2 - \overline{\lim_{n,m}} \|f_n + f_m\|_2^2.$$

Since $\|f_n\|_2 \le 1$ for all n, $\overline{\lim}_n \|f_n\|_2^2 \le 1$. So

$$\overline{\lim_{n,m}} \|f_n - f_m\|_2^2 \le 4 - \overline{\lim_{n,m}} \|f_n + f_m\|_2^2 = 4 - \lim_{n,m} \|f_n + f_m\|_2^2 = 4 - 2\cdot 2 = 0.$$

Since $\|f_n - f_m\|_2^2$ is a positive sequence, it follows that its limit exists and is 0, so $\{f_n\}$ is a Cauchy sequence. Since $L^2[0,1]$ is complete, $\{f_n\}$ converges in $L^2[0,1]$.

2. Solutions for August 2000

1. Solution. Remark: This is a thoroughly dumb problem (**a**) Obvious. Since the E_n are disjoint, for $x \in [0,1]$ the function $\chi_n(x)$ takes the value 1 at most once, and 0 the rest of the time.

(**b**) Compute

$$\|f_n\|_2 = \sqrt{\int_0^1 \left(\frac{1}{\sqrt{mE_n}}\chi_n\right)^2} = \sqrt{\frac{1}{(\sqrt{mE_n})^2}\int_{E_n} 1} = \sqrt{\frac{1}{(\sqrt{mE_n})^2}\cdot mE_n} = 1.$$

The sequence $\|f_n\|_2$ is thus constantly 1 so converges to 1.

(**c**) Fix $g \in L^2[0,1]$. Since we are on a set of finite measure, $g \in L^1[0,1]$. By monotonicity, $mE_n \le m[0,1] = 1$, so for all $x \in [0,1]$

$$|f_n(x)g(x)| \le |1/\sqrt{mE_n}\chi_n(x)| \cdot |g(x)| \le |g(x)|.$$

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[3] Alternate proof: Actually, a much stronger statement can be proved. If there exists $g \in A$ such that the H-product $f * g$ has radius of convergence 1, as the geometric does, then neither f nor g can be analytically continued to every point on the circle $|z| = 1$. Suppose WLOG f can be, so as in proof 1, $1/t > 1$. Since $g \in A$, $1/s \ge 1$. But by equation a, the H-product has radius of convergence $\frac{1}{ts} > 1$, contradicting the assumption that its radius of convergence is exactly 1.

[4] Remark: the proof will not work without using $\overline{\lim}$; there is no reason why $\lim \|f_n\|_2$ should exist

And therefore we are justified applying the LDCT

$$\lim_{n\to\infty}\int_0^1 f_n g = \int_0^1 (\lim_{n\to\infty} f_n) g = \int_0^1 0\cdot g = 0.$$

2. Solution. If $|z| > 1$, $|1/z^2| < 1$, so $z \mapsto 1 + 1/z^2$ maps the set $\{|z| > 1\}$ into $\{z \mid |z-1| < 1\}$, which is within the region where the principle branch of $\sqrt{\cdot}$ is defined:

$$\boxed{\text{Picture here}}$$

So no funny business with dogbones.

Notice that $\sqrt{1+1/z^2}$ is analytic outside of the unit disk, so we should flip it:

$$\boxed{\text{Picture here}}$$

Let

$$w = 1/z, \qquad \text{so that} dw = -\frac{dz}{z^2}.$$

Under this transformation, the curve $(1/z)\circ C$, a curve contained in the unit disk winding once CW around 0. Denote by C' this new curve in the CCW orientation, that is, $C'(t) = 1/C(-t)$. Then we get

$$\int_C z\sqrt{1+\frac{1}{z}}\,dz = \int_{1/C} \frac{1}{w}\sqrt{1+w^2}\left(-\frac{dw}{w^2}\right) = -\int_{C'} \frac{\sqrt{1+w^2}}{w^3}\,dw.$$

Then $\sqrt{1+w^2}$ is analytic and $1/w^3$ is analytic everywhere being considered except for a pole at $w = 0$. We compute

$$\frac{d^2}{dw^2}\left.\sqrt{1+w^2}\right|_0 = \frac{d}{dw}\left.\frac{1/2\cdot 2w}{\sqrt{1+w^2}}\right|_0 = \left.\frac{\sqrt{1+w^2} - w^2/\sqrt{1+w^2}}{1+w^2}\right|_0 = 1$$

and thus we have a Laurent expansion

$$\frac{\sqrt{1+w^2}}{w^3} = \cdots + \frac{1}{2!}\cdot\frac{1}{z} + \cdots,$$

so by the residue theorem

$$\int_C z\sqrt{1+\frac{1}{z}}\,dz = -\int_{C'} \frac{\sqrt{1+w^2}}{w^3}\,dw = 2\pi i\frac{1}{2} = -\pi i.$$

Remark: In my notes I have the answer as πi, but typing it up write now I convinced myself that it's $-\pi i$. So either present me or past me is making an error.

3. Solution. (**a**) The Cantor-Lebesgue function φ is an increasing function that satisfies $\varphi(0) = 0$, $\varphi(1) = 1$, and $\varphi'(x) = 0$ for almost all $x \in [0,1]$. Since it is increasing, it has bounded variation. But

$$\int_0^1 \varphi' = 0 \neq 1 = \varphi(1) - \varphi(0).$$

So φ cannot be absolutely continuous since it would have to satisfy the FTC on $[0,1]$, which it does not.

(**b**) [5] Let $\varepsilon > 0$. Since $g_{\varepsilon/2}$ is absolutely continuous, there exists δ such that for all disjoint collections of intervals $\{[a_n, b_n]\}$,

$$\sum_n |b_n - a_n| < \delta \qquad \Rightarrow \qquad \sum_n |g_{\varepsilon/2}(b_n) - g_{\varepsilon/2}(a_n)| < \varepsilon/2.$$

Further by definition of T_0^1, for all collections of disjoint subintervals of $[0,1]$,

$$\sum_n |(g_{\varepsilon/2} - f)(b_n) - (g_{\varepsilon/2} - f)(a_n)| < \varepsilon/2$$

so by the triangle inequality, $\sum_n |b_n - a_n| < \delta \Rightarrow$

$$\sum_n |f(b_n) - f(a_n)| \le \sum_n |f(b_n) - f(a_n) - (g_{\varepsilon/2}(b_n) - g_{\varepsilon/2}(a_n))| + \sum_n |g_{\varepsilon/2}(b_n) - g_{\varepsilon/2}(a_n)|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

so f is absolutely continuous.

4. Solution. (**a**) We'll write,

$$f(z) = \prod_{k=1}^{n} \varphi_k(z) \prod_{j=1}^{m} \frac{1}{\psi_j(z)}$$

where each φ_k, ψ_j is the standard form of an analytic isomorphism of the unit disk. In particular, these functions map the unit circle into itself and their only zeros are at the points a_k and b_j respectively. Let C denote the curve $t \mapsto f(e^{it})$. Then

$$\text{Winding \# of } f(e^{it}) = \frac{1}{2\pi i}\int_C \frac{1}{\zeta}\, d\zeta = \frac{1}{2\pi i}\int_{|\zeta|<1} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \mathrm{Vararg}_{|z|=1}(f),$$

where we are just using the definitions of winding number, path integral, and variation of argument. But it's clear that $\mathrm{Vararg}_{|z|=1}(f) = n - m$ since the m zeros of ψ_j become m poles of f.

(**b**) Equivalently, we are counting the roots of

$$\prod_{k=1}^{n} \varphi_k(z) - \omega \prod_{j=1}^{m} \frac{1}{\psi_j(z)} \overset{\text{call}}{=} g(z) - \omega h(z).$$

Since the the unit disk is stable under all φ_k, ψ_j, we have $|g(z)| = 1 = |h(z)|$ for all $|z| = 1$. Thus

$$|g(z) - (g(z) - \omega h(z))| = |\omega h(z)| < 1 = |g(z)|,$$

so by Rouche's theorem $g(z)$ and $g(z) - \omega h(z)$ have the same number of zeros in the unit disk, counting multiplicity. But $g(z)$ clearly has n roots counting multiplicity, so we're done.

5. Solution. I think this is one of those "keep doing what you think you should be doing and you get the answer" problems. I have the solution in my notes... don't want to type it out... god I hate Real.

6. Solution. (**a**) Remark: Standard problem, with a slight twist: you need to split up the sum due to accumulations of poles at both 0 and ∞.

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[5] The proof does not use or need the fact that f is of bounded variation

Since a compact subset of $\mathbb{C}$ is bounded, there is a natural number L so that for all $z \in K$, $|z| < L$.

$\sum_{k=L}^{\infty}$ *converges uniformly on compact subsets* For $k > \log L/\log 2+1$, so that $2^{k-1} > L$,

$$|z-2^k| > 2^k - L > 2^k - 2^{k-1} = 2^{k-1}$$

Well, here's what I have written.. this can be done better....

So for such z,

$$\left|\frac{2^k}{(z-2^k)^2}\right| < \frac{2^k}{(2^{k-1})^2} = 2^{k-2k-2} = (1/2)^{k+2}.$$

Since $\sum_{k=L}^{\infty}(1/2)^{k+2} < \infty$, by the M-test, we get the result.

$\sum_{k=-\infty}^{L}$ *converges uniformly on compact subsets* A compact set meeting the requirements "stays away from 0 and powers of 2." That is, there exists $\delta > 0$ such that for all $z \in K$, and $k \in \mathbb{Z}$,

$$|z-2^k| > \delta \qquad \text{and,} \qquad |z| > \delta$$

so... (here's just what i have written)

$$\left|\frac{2^k}{(z-2^k)^2)}\right| < \frac{2^k}{\delta^2} \implies \sum_{k=-\infty}^{0} \frac{2^k}{(z-2^k)^2}$$

converges uniformly by the Weierstrass M-test, since $\sum_{k=-\infty}^{L} \frac{2^k}{\delta} < \infty$.

Now using both steps, we conclude the entire summation converges since each part does (basically the definition).

To show F is meromorphic you just need to note that the only accumulation point of the set $\{2^k \mid k \in \mathbb{Z}\}$ is 0; if we remove 0 then there's no accumulation of singularities of the locally analytic function F, which is what it means to be meromorphic.

The final identity is just formal, every step valid by uniform convergence,

$$F(2z) = \sum_{k=-\infty}^{\infty} \frac{2^k}{(2z-2^k)^2} = \sum_{k=-\infty}^{\infty} \frac{2}{4}\frac{2^{k-1}}{(z-2^{k-1})^2} = \frac{1}{2}F(z).$$

(**b**) Suppose not. Define a function on $\mathbb{C}-\{0\}$ by

$$G(z) = \frac{1}{F(z)}.$$

This function is analytic on the domain being considered, and it has zeros exactly where F has poles (note the zero 2^k of G corresponds to the pole 2^{-k} of F. You also need to make sure that in fact F has an actual pole of order 2 at each 2^k and not a removable singularity).

I claim that 0 is a removable singularity of G. This is because

$$G(z) = \frac{1}{F(z)} = \frac{1}{2F(2z)} = \frac{1}{2}G(2z)$$

and so $G(z) = \frac{1}{2^k}G(2^k z)$. Without getting too technical, we can pick some compact neighborhood, say a large annulus, such that for every $|z| < 1$, $2^k z$ is in the compact neighborhood for some $k \geq 0$. Being continuous, G is bounded on this neighborhood, so the equality we just proved shows that G is bounded on $|z| < 1$, so $z = 0$ is a removable singularity.

But now we have a contradiction: $G(0)$ can be defined to make $G(z)$ analytic on a neighborhood of 0, but G has an accumulation of zeros at 0 corresponding to the accumulation of poles of F at ∞ (here's where it's importation we verified that F has actual poles). Thus, $G(z)$ is identically 0, which is absurd.[6]

3. Solutions for January 2001

1. Solution. (**a**) Let $\varepsilon > 0$. By absolute continuity, there is δ such that

$$\sum_n |b_n - a_n| < \delta \qquad \Longrightarrow \qquad \sum_n |f(b_n) - f(a_n)| < \varepsilon$$

E has measure 0, so there exist closed intervals $\{I_n\}$ covering E such that

$$\sum_n \ell(I_n) < \delta$$

Since f is continuous and each I_n is compact, f attains it minimum at some $a_n \in I_n$ and maximum at some $b_n \in I_n$. Then each $[f(a_n), f(b_n)]$ covers $f(I_n)$, so the collection of intervals $\{[f(a_n), f(b_n)]\}$ covers $f(E)$. Since $[a_n, b_n] \subset I_n$,

$$\sum_n |b_n - a_n| \leq \sum_n \ell(I_n) < \delta \qquad \Longrightarrow \qquad \sum_n \ell([f(a_n), f(b_n)]) < \varepsilon.$$

Since ε was arbitrary, $f(E)$ has measure 0.

(**b**) Let E be measurable. By inner regularity, there is an F_σ set F such that $F \subseteq E$ and $m(E - F) = 0$. Let $F = \bigcup_n F_n$, F_n closed. Each $F_n \subseteq [0, 1]$ and they are therefore compact, so each $f(F_n)$ is compact by continuity of f. So,

$$f(F) = f\left(\bigcup_n F_n\right) = \bigcup_n f(F_n) \qquad \Longrightarrow \qquad f(F) \text{ is } F_\sigma.$$

By part (a), since $E - F$ has measure 0, $f(E - F)$ has measure 0. Then $f(E) = f(F \cup (E - F)) = f(F) \cup f(E - F)$, so again by inner regularity, $f(E)$ is measurable.

4. Solution. Ω_c *open.* Since u is harmonic, it is continuous, therefore $u^{-1}(c, \infty)$ is open in $\mathbb{C}$.

Ω_c *nonempty.* Suppose there exists c such that $u(z) \leq c$ for all $z \in \mathbb{C}$. Since $\mathbb{C}$ is simply connected, u is the real part of an entire function f. Since

$$|e^{f(z)}| = e^{\mathrm{Re}(f(z))} = e^{u(z)},$$

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[6] There is a second (harder) proof. You can also define,

$$G(z) = \frac{1}{F(1/z)}.$$

This is what I tried doing first and it didn't work out because I couldn't show neatly that 0 was a removable singularity, but I bet it can be done. This version is different from the previous one in that you use Liouville at the end.

we have $|e^{f(z)}| \leq e^c$ for all $z \in \mathbb{C}$. But $e^{f(z)}$ is an entire function, being the composition of two entire functions, so by Liouville it is constant. So f is constant so, therefore, is u.

Ω_c *simply connected.* Let γ be a Jordan curve contained in Ω_c. Then the interior of γ, call it U, is a connected domain. Take $z_0 \in U$, and suppose $z_0 \notin \Omega_c$. Then $u(z_0) \leq c$, while for all $z \in \gamma$, $u(z) > c$ since $\gamma \subseteq \Omega_c$. But this contradicts the minimum modulus principle for harmonic functions,[7] since u is not constant. Thus $U \subseteq \Omega_c$, that is, γ is nullhomotopic.

4. Solutions for August 2001

2. Solution. Since the integrand is a rational function, it is meromorphic; its singularities are the $2n$th roots of -1. Now let $\zeta = e^{\pi i/2n}$, so that $\zeta^{2n} = -1$. Choose the path Δ_R that is composed of the straight line from 0 to R, the arc from R to $R\zeta^2 = Re^{\pi i/n}$, and the line segment from $R\zeta^2$ back to 0.

Picture here

Then Δ_R contains exactly one singularity of the integrand, ζ. Futher, since

$$\frac{d}{dz} z^{2n} + 1\bigg|_{\zeta} = 2n\zeta^{2n-1} \neq 0,$$

it follows that ζ is a simple pole. Thus the residue at ζ is

$$\operatorname{Res}_{\zeta} \frac{1}{z^{2n}+1} = \frac{1}{\frac{d}{dz} z^{2n} + 1\big|_{\zeta}} = \frac{1}{2n\zeta^{2n-1}}.$$

By the residue theorem,

$$\int_{\Delta_R} \frac{dz}{z^{2n}+1} = \pi i \left[\frac{1}{n\zeta^{2n-1}}\right] = -\frac{\pi i \zeta}{n}$$

as $\zeta^{2n} = -1$.

Now parameterize the line segments of Δ_R by $\delta_1(r) = r$, and $\delta_3(r) = r\zeta^2$, so $\delta_1'(r) = 1$, $\delta_3'(r) = \zeta^2$. Then the integrals along δ_1 and δ_2 satisfy

$$\begin{aligned}
\int_{\delta_3} \frac{dz}{z^{2n}+1} &= \int_{r=R}^{0} \frac{\zeta^2 dr}{(r\zeta^2)^{2n}+1} \\
&= \zeta^2 \int_{r=R}^{0} \frac{dr}{r^{2n}+1} = -\zeta^2 \int_{r=0}^{R} \frac{dr}{r^{2n}+1} \\
&= -\zeta^2 \int_{\delta_1} \frac{dz}{z^{2n}+1},
\end{aligned}$$

thus,

$$\begin{aligned}
\int_{\Delta_R} \frac{dz}{z^{2n}+1} &= \int_{\delta_1} \frac{dz}{z^{2n}+1} + \int_{\delta_2} \frac{dz}{z^{2n}+1} + \int_{\delta_3} \frac{dz}{z^{2n}+1} \\
&= (1-\zeta^2) \int_{\delta_1} \frac{dz}{z^{2n}+1} + \int_{\delta_2} \frac{dz}{z^{2n}+1}.
\end{aligned}$$

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[7] Here's a quick proof: again u is the real part of an analytic f, and look at e^{-f}. Then $|e^{-f}| = |e^{-u}|$. If u attains its minimum value on the interior, then e^f attains its maximum as well, so is constant.

On δ_2, $|z| = R$, and $|z^{2n}+1| \geq 2R^{2n}$. So if $n > 1$,[8]

$$\left|\int_{\delta_2} \frac{dz}{z^{2n-1}+1}\right| \leq \ell en(\delta_2) \sup_{z\in\delta_2} \left|\frac{1}{z^{2n}+1}\right| \leq \frac{R^2}{2R^{2n}} \to 0.$$

Now we simply take limits to get,

$$(1-\zeta^2)\int_{x=0}^{\infty} \frac{dx}{x^{2n-1}+1} = -\frac{\pi i \zeta}{n}$$

which simplifies down to

$$\int_{x=0}^{\infty} \frac{dx}{x^{2n-1}+1} = -\frac{\pi i}{n(\zeta^{-1}-\zeta)} = -\frac{\pi i}{n\cdot -2i\,\mathrm{Im}(\zeta)} = \frac{\pi/2n}{\sin(\pi/2n)}$$

3. Solution. Remark: there are two ways to prove the integration by parts formula. Recall that in calculus, integration by parts is proved from the product rule for derivatives, so you can try and recreate that proof. You can also prove it directly from Fubini's theorem.

Proof 1: For all $x \in [0,\infty)$,

$$|G(x)| = \left|\int_x^{\infty} g(y)\,dy\right| \leq \int_x^{\infty} |g(y)|\,dy \leq \int_0^{\infty} |g(y)|\,dy < \infty.$$

So $G(x)$ is bounded, as is $F(x)$. The product of an integrable function with a bounded function is integrable, so $f(x)G(x)$ and $g(x)F(x)$ are integrable.

4. Solution. Note that it is not possible to prove that h is continuous![9] *Step one.* Notice that since g is bounded on $|z| > 1$, it has a removable singularity at ∞. That is, $|g(1/z)|$ is bounded on the punctured disk $\{|z| < 1, z \neq 0\}$, so can be made to be analytic on $\{|z| < 1\}$.

Step two. Define $k(z) = \overline{g(1/\overline{z})}$. Then k is an analytic function on $|z| < 1$.[10] Fix $z \in C$. By the definition of h and k,

$$\lim_{r\longrightarrow 1^-} k(rz) = \lim_{r\longrightarrow 1^-} \overline{g(1/r\overline{z})} = \lim_{r\longrightarrow 1^+} \overline{g(rz)} = \overline{h(z)}$$

since $1/\overline{z} = z$.

Step three. Let $u(z) = \mathrm{Re}(f(z)-k(z))$ and $v(z) = \mathrm{Im}(f(z)+k(z))$. Then u and v are the real and imaginary parts of analytic functions, so are harmonic function on $|z| < 1$. Then

$$\lim_{r\longrightarrow 1^-} u(rz) = \mathrm{Re}(\lim_{r\longrightarrow 1^-} f(z)-k(z)) = \mathrm{Re}(h(z)-\overline{h(z)}) = 0$$

and

$$\lim_{r\longrightarrow 1^-} v(rz) = \mathrm{Im}(\lim_{r\longrightarrow 1^-} f(z)+k(z)) = \mathrm{Im}(h(z)+\overline{h(z)}) = 0.$$

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[8] $n = 1$?

[9] example

[10] the easiest way to see this is to show k satisfies the Cauchy-Riemann equations

So both u and v can be extended continuously to be 0 on the unit circle. By the maximum and minimum modulus principles for harmonic functions,

$$u \equiv 0 \qquad \text{and} \qquad v \equiv 0$$

so

$$\operatorname{Re}(f(z)) = \operatorname{Re}(k(z)) = \operatorname{Re}(g(1/\overline{z}) \qquad \text{and} \qquad \operatorname{Im}(f(z)) = -\operatorname{Im}(k(z)) = \operatorname{Im}(g(1/\overline{z})$$

And thus $f(z) = g(1/\overline{z})$ on $|z| < 1$. If g is nonconstant, $g(1/\overline{z})$ is not an analytic function since it fails Cauchy-Riemann,[11] contradicting the fact that f is analytic. Thus g is constant, and hence so is f.

6. Solution. *Step one.* Suppose f has an infinite number of zeros in $|z| < 1$. By Bolzano-Weierstrass, they must have an accumulation point. They cannot accumulate to the boundary, given the condition that f is arbitrarily large near the boundary. They cannot accumulate to the interior, since then f would be identically zero. Thus f has finitely many zeros (of finite order).

Step two. Let the zeros of f be $z_0, \ldots, z_n$ with orders $m_0, \ldots, m_n$. Then,

$$g(z) = \frac{\prod_{k=0}^{n}(z - z_k)^{m_k}}{f(z)}$$

is an analytic function in $|z| < 1$. But

$$\lim_{|z| \longrightarrow 1^-} |g(z)| = \lim_{|z| \longrightarrow 1^-} \frac{\prod_{k=0}^{n}(z - z_k)^{m_k}}{f(z)} = 0.$$

By the maximum modulus principle, g is identically zero, which means that f is not finite on $|z| < 1$. $\Rightarrow\!\Leftarrow$

5. Solutions for January 2002

1. Solution. $L^2[0,1]$ is a Hilbert space with inner product

$$(f, g) = \int_0^1 fg$$

To say the sequence $\{\varphi_n\}$ is orthonormal means

$$(\varphi_i, \varphi_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

So

$$(\sigma_n(x))^2 = \frac{1}{n^2}\left[\sum_{i=1}^{n} \varphi_i^2(x) + 2\sum_{i \neq j} \varphi_i(x) \cdot \varphi_j(x)\right]$$

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[11] Another argument: $g(1/\overline{z})$ is not analytic because g is conformal (a.e.) while $1/\overline{z}$ is anti-conformal (a.e.), so their composition is anti-conformal.

and

$$\begin{aligned}\|\sigma_n(x)\|_{L^2}^2 &= \frac{1}{n^2}\int_0^1 \left[\sum_{i=1}^n \varphi_i^2(x) + 2\sum_{i\neq j}\varphi_i(x)\cdot\varphi_j(x)\right]\\ &= \frac{1}{n^2}\sum_{i=1}^n(\varphi_i,\varphi_i) + \frac{2}{n^2}\cancelto{0}{\sum_{i\neq j}(\varphi_i,\varphi_j)}\\ &= \frac{1}{n^2}\cdot n \xrightarrow{n\to\infty} 0.\end{aligned}$$

Thus $\sigma_n \to 0$ in $L^2([0,1])$. Since $\|f\|_{L^1} \leq c\cdot\|f\|_{L^2}$ for some constant c, $\sigma_n \to 0$ in $L^1([0,1])$. Convergence in mean implies convergence in measure, so we are done.

6. Solutions for August 2002

1. Solution. (**a**) Let $\varepsilon > 0$. There is a $\delta_1 > 0$ such that if $\{[c_n, d_n]\}$ is any collection of subintervals of $[f(a), f(b)]$,

$$\sum |c_n - d_n| < \delta_1 \qquad \Rightarrow \qquad \sum |g(c_n) - g(d_n)| < \varepsilon$$

by absolute continuity of g. By absolute continuity of f, there is a $\delta_2 > 0$, depending on ε,δ_1, such that

$$\sum |a_n - b_n| < \delta_2 \qquad \Rightarrow \qquad \sum |f(a_n) - f(b_n)| < \delta_1.$$

Since f is strictly increasing, it maps intervals to intervals. Thus if $\{[a_n, b_n]\}$ is a collection of subintervals of $[a, b]$, then $\{[f(a_n), f(b_n)]\}$ is a collection of subintervals of $[f(a), f(b)]$ and

$$\sum |a_n - b_n| < \delta_2 \qquad \Rightarrow \qquad \sum |f(a_n) - f(b_n)| < \delta_1 \qquad \Rightarrow \qquad \sum |g(f(a_n)) - g(f(b_n))| < \varepsilon.$$

So $g \circ f$ is absolutely continuous.

(**b**) An absolutely continuous function is differentiable almost everywhere. Further, a function is differentiable at a point only if it is differentiable on an open interval containing that point. Thus at all points where $g \circ f$ is differentiable, we can just use the normal chain rule from calculus.

(**c**) Since f is strictly increasing, is derivative is bounded on $[a, b]$. The product of an integrable function with a bounded function is integrable, so $\varphi(f(x)) \cdot f'(x)$ is integrable. Let $g(z) = \int_{f(a)} x\varphi(y)\,dy$. Then g is absolutely continuous. So by part (b),

$$(g \circ f)'(x) = \frac{d}{dx}\int_{f(a)}^{f(x)} \varphi(y)\,dy = \varphi(f(x)) \cdot f'(x).$$

By part (a), since $g \circ f$ is absolutely continuous, the FTC applies, so integrating both sides gives

$$\int_a^b \varphi(f(x))\cdot f'(x)\,dx = \int_a^b (g\circ f)'(x)\,dx = g(f(b)) - g(f(a)) = \int_{f(a)}^{f(b)} \varphi(y)\,dy - \cancelto{0}{\int_{f(a)}^{f(a)} \varphi(y)\,dy}$$

4. Solution. Since g is entire and has no zeros, the function g'/g is entire. Pick any point $z_0 \in \mathbb{C}$ and set $w = g(z_0)$, $w \neq 0$. For any $z \in \mathbb{Z}$, choose a path $\gamma : z_0 \to z$ and define,

$$f(z) = \int_\gamma \frac{g'(\zeta)}{g(\zeta)}\, d\zeta.$$

since $\mathbb{C}$ is simply connected, by the homotopy form of Cauchy's theorem, f is a well defined function, and as it's the antiderivative of the entire function g'/g, it is also entire.

Now compute,

$$\frac{d}{dz}\exp(f(z)) = \frac{g'(z)}{g(z)}\exp(f(z))$$

so

$$\frac{d}{dz}\frac{\exp(f(z))}{g(z)} = \frac{\frac{g'(z)}{g(z)}\exp(f(z)) \cdot g(z) - g'(z)\exp(f(z))}{(g(z))^2} = 0.$$

Thus the entire function $\frac{\exp(f(z))}{g(z)}$ is constant. What constant? Well, by Cauchy's theorem $f(z_0) = 0$, so

$$\frac{\exp(f(z_0))}{g(z_0)} = \frac{1}{w}$$

thus

$$g(z) = w\exp(f(z)),$$

and if we let ω be any complex logarithm of w, we get that $\omega \cdot f$ is the required function.[12]

6. Solution. **(a)** Let $|q'(0)| = \alpha < 1$. Choose $\varepsilon > 0$ such that $1 > \alpha + \varepsilon \overset{\text{call}}{=} \beta$. Since q is differentiable at 0, there exists $\delta > 0$ such that

$$|z| < \delta \qquad \Rightarrow \qquad \left|\frac{q(z) - q(0)}{z - 0}\right| = \left|\frac{q(z)}{z}\right| < \alpha + \varepsilon = \beta$$

by the definition of differentiability. So q maps the disk $|z| < \delta$ into itself and $|q(z)| < \beta \cdot |z|$. Thus

$$|q^{(n)}(z)| < \beta^n \cdot |z| < \beta^n \cdot \delta.$$

Since $\beta < 1$, the geometric series $\sum \beta^n \cdot \delta$ is finite.[13] Thus the tails of the series converge to zero, so

$$\|S_n - S_m\| = \sup_{|z|<\delta}\left|\sum_{k=m+1}^{n} q^{(k)}(z)\right| \leq \sup_{|z|<\delta}\sum_{k=m+1}^{n}|q^{(k)}(z)| \leq \sum_{k=m+1}^{n}\beta^k \cdot \delta \xrightarrow{n\to\infty} 0.$$

Therefore the sequence is uniformly Cauchy, hence converges uniformly to an analytic function on the disk $|z| < \delta$.

(b) *Step 1.* Suppose there exists r such that on $|z| < r$,

$$\sup_{|z|<r}|S_n(z)| \xrightarrow{n\to\infty} \infty.$$

---

[12] Remark: This is a standard complex problem. It's done in Lang. Whatever your solution is, it is important that is uses the topology of the complex plane! Pick a different Riemann surface, say the punctured complex plane. The function $g(z) = z$ is nonzero on the entire punctured complex plane; however it does not have a well defined logarithm. The point is that you need simply connectedness.

[13] Therefore S_n converges uniformly to an analytic function by the Weierstrass M-test. What follows is basically a proof of the theorem

That is, the sequence of uniform norms $\|S_n\|$ is bounded. Thus by Bolzano-Weierstrass there is a subsequence, say $\|S_{n_j}\|$, that is Cauchy. This subsequence of functions therefore converges uniformly on the disk $|z| < r$ to some analytic function, say h, and further $S'_{n_j} \to h'$.

Step 2. By the chain rule,

$$(q^{(k)})'(0) = q'(q^{(k-1)}(0)) \cdots q'(q(0)) \cdot q'(0) = (q'(0))^k,$$

since $q(0) = 0$. So,

$$S'_{n_j}(0) = \sum_{k=1}^{n_j} (q^{(k)})'(0) = \sum_{k=1}^{n_j} (q'(0))^k.$$

Since $|q'(0)| > 1$, the sequence $S'_{n_j}(0) \to \infty$. But $S'_{n_j}(0) \to h'(0)$, which must be a finite number. $\Rightarrow\Leftarrow$

7. Solutions for August 2003

6. Solution. Suppose otherwise, $|z| = 1 \Rightarrow |P(z)| < 1$. Then for all $|z| = 1$,

$$1 = |z^n| > |P(z)| = |(z^n - P(z)) - z^n|$$

By Rouche's theorem, since z^n, P are analytic on $|z| < 1$, z^n and $z^n - P(z)$ have the same number of zeros counting multiplicity inside the disk. But 0 is a zero of multiplicity n of z^n, while $z^n - P(z) = a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial of degree $(n-1)$, so has at most $(n-1)$ roots counting multiplicity. $\Rightarrow\Leftarrow$

8. Solutions for January 2004

6. Solution. (**a**) Let $\mathcal{G}$ be the family of maps $\{\varphi \circ h\}$, where $\varphi(z) = \frac{\sqrt{z}-1}{\sqrt{z}+1}$.

Picture here

First, notice that φ is an analytic isomorphism from $\mathbb{C} - (-\infty, 0] \to \mathbb{D}$. Therefore, $\mathcal{G}$ is a normal family if and only if $\mathcal{H}$ is.[14] But $\mathcal{G}$ is a normal family because it is uniformly bounded by 1.

(**b**) Let $h_n \in \mathcal{H}$ such that $h_n(0) \to 0$. Let $g_n = \varphi \circ h_n$.

Claim: $g_n \to -1$ uniformly on compact subsets of $\mathbb{D}$.

Pf: Suppose not. Then there is a compact set $K \subset \mathbb{D}$, $\varepsilon > 0$, and subsequence g_{n_k} such that,

$$\sup_{z \in K} |g_{n_k}(z) + 1| > \varepsilon.$$

But g_{n_k} is itself a normal family, so there is a subsequence $g_{n_{k_j}}$ that converges uniformly on K to some analytic function $g : \mathbb{D} \to \mathbb{D}$. By continuity, $\lim_{n\to\infty} g_n(0) = \varphi(0) = -1$, so $g(0) = -1$. Since g maps into the unit disk, by the maximum modulus principle, g must be identically -1. So $g_{n_{k_j}}$ converges uniformly to -1, that is, there is a J such that for all $j \geq J$,

$$\sup_{z \in K} |g_{n_{k_j}}(z) + 1| < \varepsilon$$

---

[14] ??why?

contradicting the fact that the whole sequence g_{n_k} is uniformly bounded away from -1.

So g_n converges to -1 uniformly, and composing with φ^{-1}, it follows that h_n converges to 0 uniformly.[15]

9. Solutions for August 2004

2. Solution. This is a standard qual problem. The answer is

$$\frac{\pi/9}{\sin \pi/9}.$$

Years where this problem appears:
August 2008 number 2
August 2004 number 2
August 2001 number 2

3. Solution. (**a**) Since $f_n \in C^1[0,1]$, the FTC holds, so for all $x \in [0,1]$,

$$f_n(x) = \int_0^x f_n' \qquad \text{by (a)}$$

Now by (b) all the derivatives f_n' are dominated by a function in $L^1[0,1]$ [by basic calc], so $f_n' \in L^1[0,1]$ and we can use the LDCT: since $f_n' \to h$ by (c),

$$\lim_{n \longrightarrow \infty} f_n(x) = \lim_{n \longrightarrow \infty} \int_0^x f_n' = \int_0^x h \overset{call}{=} f(x).$$

By the Lebesgue Differentiation Lemma, $f(x)$ is an absolutely continuous function. Therefore all that remains to show is that the convergence is uniform.

(**b**) Since both $f_n' \to h$ pointwise and $\|f_n'\|_1 \to \|h\|_1$, it follows that $f_n' \to h$ in $L^1[0,1]$ as well.[16] That is, given $\varepsilon > 0$, $\int_0^1 |f_n' - h| < \varepsilon$ for sufficiently large n, so

$$|f_n(x) - f(x)| = \left| \int_0^x f_n' - \int_0^x h \right| = \left| \int_0^x f_n' - h \right| \leq \int_0^x |f_n' - h| \leq \int_0^1 |f_n' - h| < \varepsilon$$

Since the inequality does not depend on x, the convergence is uniform by definition.

4. Solution. (**a**) Let $u : \mathbb{C} \to \mathbb{R}$ be harmonic. Define a complex function

$$f = u_y + iu_x.$$

That is, f is basically the gradient of u. Since u is harmonic it is C^2, f is continuous with continuous real partials. Further,

$$(u_y)_x = (u_x)_y \qquad \text{since } u \text{ is } C^2$$
$$(u_y)_y = -(u_x)_x \qquad \text{since } \nabla^2 u = 0.$$

Therefore by the converse to Cauchy-Riemann, f is an entire function.

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[15] ??why?

[16] cite

(**b**) Now pick any $\omega \in \mathbb{C}$. Then $\omega \cdot f$ is also an entire function, and nonconstant if ∇u is nonconstant. By Picard's theorem, it misses at most one point, in particular it hits the imaginary axis. Thus, for some z, $\operatorname{Re}(\omega f(z)) = 0$. But,

$$\operatorname{Re}(\omega f(z)) = \operatorname{Re}\omega \cdot u_y(z) - \operatorname{Im}\omega \cdot u_x(z).$$

Thus for this z,

$$\frac{\operatorname{Re}\omega}{\operatorname{Im}\omega} = \frac{u_x(z)}{u_y(z)}.$$

The thing on the left is the slope of ω, and the thing on the right is the slope of ∇u at z.

5. Solution. (**a**) Notice that $f_n(x) \geq 0$ for all x, n. Suppose that $f_n(x) \leq f_{n-1}(x)$, the base case already assumed. Then

$$\begin{aligned} f_{n+1}(x) = \left(\int_0^x f_n\right)^{1/2} &\leq \left(\int_0^x f_{n-1}\right)^{1/2} \qquad \text{since } f_n \text{ nonnegative} \\ &= f_n(x). \end{aligned}$$

Thus by induction, $\{f_n(x)\}$ is monotonically decreasing and a nonnegative sequence. Therefore it has a nonnegative limit, namely its infimum.

(**b**) Since $f_0 \in L^1$ and for all $x \in [0,1]$, $f_n(x) \leq f_0(x)$, by the LDCT

$$\lim_{n\to\infty} \int_0^x f_n = \int_0^x \lim_{n\to\infty} f_n = \int_0^x f.$$

By continuity of $\sqrt{\cdot}$,

$$\left(\int_0^x f\right)^{1/2} = \lim_{n\to\infty} \left(\int_0^x f_n\right)^{1/2} = \lim_{n\to\infty} f_{n+1}(x) = f(x).$$

(**c**) Part (b) says that f is integrable, since its integral on $[0,1]$ is just $(f(1))^2$. Now $\int_0^x f$ is absolutely continuous by the Lebesgue lemma so in particular it is continuous. So $f(x) = \sqrt{\int_0^x f}$ is continuous. By the **old** FTC from elementary calculus if f is continuous, then $\int_0^x f$ is C^1 and $d/dx \int_0^x f = f(x)$, everywhere.[17] But also $\int_0^x f = (f(x))^2$, so by the chain rule

$$f(x) = d/dx \int_0^x f = 2f'(x)f(x)$$

Thus if $f(x) > 0$ we cancel $f(x)$ from both sides to get $f'(x) = \frac{1}{2}$.

(**d**) The solution to this differential equation with the constraint $f(x) > 0$ is $f(x) = \frac{x}{2} + c$ for some $c \geq 0$.

6. Solution. The map $z \overset{\varphi}{\mapsto} \frac{z+1}{z-1}$ is an analytic isomorphism from the right half plane to the unit disk.

Picture here

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[17] This may seem like much, but I think you must say all this... all the theorems from analysis are true almost everywhere, and this problem wants you to prove it everywhere. In particular, the lebesgue differentiation lemma wouldn't work

Let $g = \varphi \circ f$. Then g maps $\mathbb{D}$ to $\mathbb{D}$, and $g(0) = \varphi(f(0)) = \varphi(1) = 0$. So by the Schwarz lemma for all $z \in \mathbb{D}$, $|z| \geq |g(z)|$. Since $f(z) = \varphi^{-1}(g(z)) = \frac{1+g(z)}{1-g(z)}$, we get

$$|f(z)| = \left|\frac{1+g(z)}{1-g(z)}\right| \leq \frac{1+|z|}{|1-g(z)|}$$

by the triangle inequality, and since $1 \geq |z| \geq |g(z)|$, by the reverse triangle inequality, $|1 - g(z)| \geq 1 - |z|$, so

$$|f(z)| \leq \frac{1+|z|}{|1-g(z)|} \leq \frac{1+|z|}{1-|z|}.$$

Remark: a better but similar problem from the Michigan qual: if f maps $\mathbb{D}$ into $\{z \in \mathbb{C} \mid -1 < \operatorname{Re} z < 1\}$, and $f(0) = 0$, show for all $z \in \mathbb{D}$,

$$|\operatorname{Im} f(z)| \leq \frac{2}{\pi} \log \frac{1+|z|}{1-|z|}.$$

10. Solutions for January 2005

1. Solution. Suppose that such a sequence does not exist. Then there exists $n \in \mathbb{N}$ such that for all $x \geq n$, $|xf(x)| > \frac{1}{n}$, since otherwise, we can pick such an x_n for each n, and this sequence will satisfy the required property. So for all $x \geq n$, $|f(x)| \geq \frac{1}{nx}$. But by comparison,

$$\int_n^\infty |f| \geq \int_n^\infty \frac{1}{nx} = \infty$$

since $\frac{1}{x} \notin L^1(n, \infty)$ for any n, contradicting the fact that $f \in L^1(0, \infty)$.

2. Solution. (**a**) Choose the contour integral below:

Picture here

and select the branch of $\sqrt{\cdot}$ such that along γ_1, $\log(z - z^2)$ has a small imaginary part. Specifically , write

$$\log(z - z^2) = \log_1 z + \log_2(1 - z)$$

and pick $\log_i$ defined by $\arg_i$ where

$$\arg_1 z \in (-2\pi, 0) \qquad \arg_2 z \in (-\pi, \pi).$$

Then by continuity, as γ_1 approaches the real axis, and $z \in \gamma_1$, $\log_1 z$ approaches $\ln|z|$ and $\log_2(1 - z)$ approaches $\ln|1 - z|$, so $\sqrt{z - z^2}$ approaches $\exp(\frac{1}{2}(\ln|z| + \ln|1 - z|))$, which will be exactly $\sqrt{x - x^2}$ on the real axis. Along γ_3, things are different. For $z \in \gamma_3$, $\log_1 z$ approaches $\ln|z| - 2\pi i$ and $\log_2(1 - z)$ approaches $\ln|1 - z|$. So $\sqrt{z - z^2}$ approaches $\exp(\frac{1}{2}(\ln|z| + \ln|1 - z|) - \pi i) = -\exp(\frac{1}{2}(\ln|z| + \ln|1 - z|))$. Thus, as γ_1, γ_3 approach the real axis, taking into account the reverse orientation of γ_3, both $\int_{\gamma_1}$ and $\int_{\gamma_3}$ approach the real integral we are trying to compute.

(**b**) Our choice of $\sqrt{\cdot}$ is clearly locally analytic, but it might not be well defined: the only way this would happen is if the function is not continuous as we loop around the slit. But as we follow γ, by the time we've made a loop we've picked up a $-2\pi i$ from $\log_2$ and a $+2\pi i$ from $\log_1$, which end up canceling. Thus $\sqrt{z - z^2}$ is analytic $\mathbb{C} - [0, 1]$.

(**c**) We now compute $\int_\gamma$. Let $w = \frac{1}{z}$. Then $dz = -\frac{1}{w^2}\,dw$, and the integral becomes

$$\int_\gamma \frac{\sqrt{z-z^2}}{z+2}\,dz = \int_{\gamma'} -\frac{1}{w^2}\frac{\sqrt{\frac{1}{w}-\frac{1}{w^2}}}{\frac{1}{w}+2}\,dw = \int_{-\gamma'} \frac{\sqrt{w-1}w^2}{w(1+2w)}\,dw$$

where γ' is oriented in the wrong (CW) direction. Then the integrand has isolated singularities at $w = 1\frac{1}{2}$ and $w = 0$.

4. Solution. f also defines a complex function,[18] which by factoring,

$$f(z) = \frac{1}{(1+z^2)+z^4(1+z^2)} = \frac{1}{(1+z^2)(1+z^4)},$$

we see that f is meromorphic with precisely six simple poles: at $\pm i$ and the fourth roots of -1.

Picture here

It is a (nontrivial) fact that the radius of convergence of $\sum_n a_n(z-2)^n$ is also the radius of the largest disk centered at 2 on which f is analytic.[19] Since the nearest singularities of f to 2 are obviously $\pm e^{i\pi/4} = \sqrt{2}/2 + (\sqrt{2}/2)i$, and

$$|2-\sqrt{2}/2+(\sqrt{2}/2)i| = \sqrt{(2-\sqrt{2}/2)^2+\frac{1}{2}} = \sqrt{5-2\sqrt{2}}$$

it follows that the radius of convergence is $\sqrt{5-2\sqrt{2}}$. By Cauchy-Hadamard,

$$\overline{\lim_{n\to\infty}}\ |a_n|^{1/n} = \text{radius of convergence} = \sqrt{5-2\sqrt{2}}.$$

6. Solution. This problem would take a couple pages to get all the details right. So for right now I'm going to leave most of them out. Here's how you should think of the problem.

Simplify the region. Think of regions on the complex plane as stretchy regions on a sphere. So this region D is just all of the sphere with two circles removed. Well if you think about it, that's just an annulus (almost). In fact you can find the map explicitly, but I'll leave the details out:

> *There is a map $\varphi : \mathbb{C} \to \mathbb{C}$ that maps D isomorphically to a region $\widetilde{D}$ which is the unit disk with a smaller disk cut out of the interior. The map φ is a bilinear transformation.*

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[18] without making a fuss, we'll just prove the required fact for the complex function f. If you want to be thorough, you need to observe that our new complex function $f(z)$ will restrict to be the original function $f(x)$ on the real line; by uniqueness, its power series will be the same as the power series for the original (real) function, and therefore have real coefficients, and also anything we prove for the coefficients (radius of convergence) of $f(z)$ will also be true of the coefficients (resp. radius of convergence) of $f(x)$. But why bother saying all that?

[19] Let's talk about this a bit. Since the problem is pretty trivial except for this fact, you are perhaps supposed to know how to prove it.

Note this statement is false for real analytic functions. The simplest example is $f(x) = 1/(x^2+1)$; it is analytic at 0 and you can compute the radius of convergence of its power series to be 1. However, f is analytic on the entire real line, even though it cannot be represented by the power series centered at 0 outside of $|x| < 1$.

Insert proof here.

Extend the map. Notice we've reduced the problem to showing that any function $\widetilde{f}$, analytic and 1:1 on $\widetilde{D}$ and satisfying $\widetilde{f}(\widetilde{D}) = \widetilde{D}$, is a bilinear transformation (set $\widetilde{f} = \varphi \circ f \circ \varphi^{-1}$). To do this we'll extend to a map on the entire unit disk. First, since $\widetilde{f}$ is analytic and 1:1, it "takes the boundary to the boundary." So by composing with a proper bilinear map, we may assume that it takes the outer circle to the outer circle. It is a theorem that we can extend the map continuously to the inner circle (and it will be an injection there). The circle is an analytic arc and we can therefore apply Schwarz reflection to get an analytic, 1:1 map on the unit disk with a *smaller* disk removed.

[Now here's where you get bogged down in details.] As you apply Schwarz reflection ad infinitum, you get an analytic, 1:1 map on the punctured unit disk (the puncture won't be in the center). Finally, we note that this puncture is obviously a removable singularity, since we're mapping inside the unit disk and therefore bounded.[20]

Classify the map. Therefore we reached our goal: we've found a map $\widetilde{g}$ which is an analytic isomorphism of the unit disk. All analytic isomorphisms of the unit disk are bilinear transformations.[21] Therefore $\widetilde{g}$ is a bilinear transformation. Its restriction to $\widetilde{D}$ is $\widetilde{f}$ and by uniqueness of complex analytic functions, $\widetilde{f}$ is also bilinear.

11. Solutions for August 2005

1. Solution. Since f is bounded, by replacing it with $f/\|f\|_\infty$ we may assume WLOG that $|f(x)| \leq 1$, since f is integrable if and only if $f/\|f\|_\infty$ is. Define the set

$$E_n = \{x \mid \frac{1}{n^4} \geq |f(x)| > \frac{1}{(n+1)^4}\}.$$

---

[20] To get a good grasp of this problem, try the following problem with is basically the same thing:

Regions in the complex plane have an equivalence relation: we call $U, V \subset \mathbb{C}$ conformally equivalent if there's an analytic isomorphism $\varphi : U \to V$. The Riemann mapping theorem can be interpreted as saying all simply connected domains are in the same conformal equivalence class.

> Fact: Two annuli are conformally equivalent if and only if the ratio
>
> $$\frac{\text{big radius}}{\text{small radius}}$$
>
> is equal for both annuli. Also, all analytic automorphisms of annuli are bilinear transformations.

[21] There are **three** objects whose analytic automorphisms you must memorize:

- All analytic isomorphisms $f : \mathbb{C} \to \mathbb{C}$ are linear ($f(z) = az + b, a, b \in \mathbb{C}$). This is a corollary of Casorati-Weierstrass.
- All analytic isomorphisms $f : \{\text{upper half plane}\} \to \{\text{upper half plane}\}$ are given by
  $$f(z) = \frac{az+b}{cz+d}$$
  where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$.
- All analytic isomorphisms $f : \{\text{unit disk}\} \to \{\text{unit disk}\}$ are given by
  $$f(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$$
  where $|\alpha| < 1$. This is a corollary of the Schwarz Lemma.

Then $E_n \subseteq \{x||f(x)| > (n+1)^{-4}\}$, so $mE_n \leq C/\sqrt{(n+1)^{-4}} = C(n+1)^2$. Further on the set E_n, $|f(x)| \leq 1/n^4$, so

$$\int_{E_n} |f| \leq \frac{1}{n^4} C(n+1)^2 = C(\frac{1}{n^2} + \frac{2}{n^3} + \frac{1}{n^4}).$$

The disjoint union of all the E_n is the support of f, so by the Monotone Convergence Theorem,

$$\int_{\mathbb{R}} |f| = \int_{\cup_n E_n} |f| = \sum_n \int_{E_n} |f| \leq \sum_n C(\frac{1}{n^2} + \frac{2}{n^3} + \frac{1}{n^4}) < \infty,$$

since the three series on the right converge by the p-test. So f is integrable.

2. Solution. (**a**) Let a_n be the coefficient in the power series. Then

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(3n+3)!}{(3n)!} \cdot \frac{(2n+2)!}{(2n)!} \cdot \frac{n)!}{(n+1)!} \cdot \frac{(4n)!}{(4n+4)!} \\ &= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)(2n+2)(2n+1)}{(n+1)(4n+4)(4n+3)(4n+2)(4n+1)} \\ &= \frac{3^3 \cdot 2^2}{4^4} = (3/4)^3.\end{aligned}$$

By the ratio test, this power series converges for all z such that $|z| < (3/4)^{-3} = (4/3)^3 = R$.

(**b**) Let $|z| = R = (4/3)^3$ Notice that if $k > n$,

$$(kn)! \geq \underbrace{(kn)(kn-1)\cdots(kn-n+1)}_{n \text{ factors}} \geq (kn-n)^n,$$

while clearly $(kn)^n \geq (kn)!$. So

$$a_n \geq \frac{(2n)^n (n)^n}{(n)^n (4n)^n} = \left(\frac{1}{2}\right)^n$$

and so,

$$|a_n z^n| = a_n |z|^n \geq \left(\frac{1}{2}\right)^n \left(\frac{4^3}{3^3}\right)^n = \left(\frac{32}{27}\right)^n \xrightarrow{n\to\infty} \infty.$$

So the series $\sum_n a_n z^n$ cannot converge since the modulus of the terms do not converge to 0.

3. Solution. Let $G(x) = \int_0^x |f(t)|^2\, dt$. By the Lebesgue lemma, $G'(x) = |f(x)|^2$. Now suppose that $G(b) > 0$, $b \geq a$, where a is given in the hypothesis. Since $|f(x)|^2 \geq 0$, G is increasing, so $G > 0$ on the set $[b, \infty)$. Therefore $1/G$ is a continuous function defined on $[b, \infty)$, with derivative satisfying

$$\left(\frac{1}{G}\right)'(x) = -\frac{G'(x)}{G^2(x)} = -\frac{|f(x)|^2}{(\int_0^x |f(t)|^2)^2} \leq -1$$

by hypothesis. But it is not possible to have a positive continuous function decreasing more than linearly.[22] So we get that for all $b \geq a$,

$$G(b) = 0 = \int_0^b |f(t)|^2 \, dt$$

Since b was arbitrary and $|f(t)|^2$ is nonnegative, by Fatou's lemma $\int_0^\infty |f(t)|^2 \, dt = 0$. Hence $|f(t)|^2 = 0$ almost everywhere, so the same must be true of $f(t)$.

4. Solution. We first look at the image of the domain $\mathbb{D}$ under the map $z \to 4/z$. This Mobius transformations sends circles through the origin to lines and all other circles to circles. So the image of the circles $|z-2| = 2$ and $|z-1| = 1$ will be some lines. Notice that $z \to 4/z$ takes the real line to itself. Further, it is a conformal map at $z = 1$ and $z = 2$. Since the circles $|z-2| = 2$ and $|z-1| = 1$ are perpendicular to the real line, their images will be perpendicular to the real line, i.e., be the vertical lines $x = 4/4 = 1$ and $x = 4/2 = 2$, resp.

Picture here

Since each C_n gets mapped to a circle, call it C_n', and $4/z$ is conformal except at 0, each C_n preserves its tangency to the circles $|z-2| = 2$ and $|z-1| = 1$, so each C_n' will be a circle tangent to the vertical lines $x = 1$ and $x = 2$, so we get the picture above.

Let $s = 4/t$. The exterior of the circle $|z| = t$ gets mapped to the interior of the circle $|z| = s$ under the map $8/z$. So if we let $N_0(s)$ be the number of C_n' contained in the interior of the circle $|z| = s$, we get $N(t) = N_0(s)$.

Now let $B_1(s)$ and $B_2(s)$ be the regions below, specifically,

$$B_1(s) = \{x + iy \mid |y| \leq \sqrt{s^2 - 4}\}$$

and

$$B_2(s) = \{x + iy \mid |y| \leq s\}.$$

If $N_1(s)$ and $N_2(s)$ represent the number of C_n' contained in $B_1(s)$ and $B_2(s)$, we have the inequality

$$N_1(s) \leq N_0(s) \leq N_2(s)$$

Now each C_n' has diameter 1 and $B_2(s)$ has height $2s$. It isn't too hard to see that $B_2(s)$ contains exactly $2[|s - \frac{1}{2}|] + 1$ circles, so we get the inequality

$$2s - 1 \leq N_2(s) \leq 2s$$

Picture here

5. Solution. (**a**) No. as a counterexample, $x \sin(1/x)$ is absolutely continuous on $[\varepsilon, 1]$ for all $\varepsilon > 0$ and continuous at 0, but it is not absolutely continuous on $[0, 1]$.

---

[22] By the mean value theorem, there is a $c \in [b, b+n]$ such that

$$\left(\frac{1}{G}\right)'(c) = \frac{(1/G)(b+n) - (1/G)(b)}{(b+n) - b} \geq \frac{-(1/G)(b)}{n} \xrightarrow{n \to \infty} 0$$

which contradicts the bound on the derivative.

(**b**) Yes. Suppose f is of bounded variation on $[0,1]$. Then f' exists almost everywhere and is integrable. In particular, for every $\varepsilon > 0$, there exists $\delta >$ such that,

$$mE < \delta \qquad \Rightarrow \qquad \int_E |f'| < \varepsilon.$$

Let $\{[a_i, b_i]\}_i$ be a countable collection of disjoint intervals such that $\sum_i |b_i - a_i| < \delta$

MISSING

6. Solution. Pf 1: Since $|z| \to 0 \Rightarrow |z|^{1/2} \to 0 \Rightarrow |f(z)| \to 0$, f has a removable singularity at $z = 0$, so WLOG $f(0) = 0$. Let $z_0 \in \mathbb{C}$, $|z_0| = R$. Then by Cauchy's formula, if $C_{2R} = \{z \mid |z| = 2R\}$,

$$|f'(z_0)| = \frac{1}{2\pi}\left|\int_{C_{2R}} \frac{f(z)}{(z-z_0)^2}\,dz\right| \leq \frac{1}{2\pi}\cdot 2\pi(2R)\cdot \sup_{|z|=2R}\frac{|f(z)|}{|z-z_0|^2}.$$

Since $|z - z_0| \geq R$ for $|z| = 2R$, and given $|f(z)| \leq |z|^{1/2}$, we get

$$|f'(z_0)| \leq 2\frac{R}{R^2}\cdot R^{1/2} = \frac{2}{R^{1/2}} \to 0$$

as $R \to \infty$. Thus f' is bounded. But f being entire implies f' is entire, so by Liouville, f' is a constant. Thus f is linear. But no nontrivial linear function can satisfy the given growth condition on $|f(z)| \leq |z|^{1/2}$, so we must have $f \equiv 0$.[23]

Pf 2: One can show that f has to be linear in another manner. Again, f has a removable singularity at 0, so WLOG $f(0) = 0$. If $|z| = 1$, $|f(z)| \leq |z|^{1/2} = 1$. Thus $f(\mathbb{D}) = \mathbb{D}$ where $\mathbb{D}$ is the unit disk, so by Schwarz's lemma, $z \in \mathbb{D} \Rightarrow |f(z)| \leq |z|$. By basic algebra, $|z| \geq 1 \Rightarrow |f(z)| \leq |z|$. Now let $g(z) = f(z)/z$. Since f is continuous and $f(0) = 0$, g is entire. Further we have shown that g is bounded by 1 on the complex plane, so by Liouville, g is constant, i.e., f is linear. Again, this contradicts the growth condition on f unless it is identically 0.

12. Solutions for January 2006

1. Solution. The function

$$\frac{x}{n}\chi_{[0,n]}$$

attains its maximum at $x = n$, at which point its value is $n/n = 1$. Thus for all x, $|\frac{x}{n}\chi_{[0,n]}| \leq 1$, so

$$\left|\frac{x}{n}\chi_{[0,n]}f(x)\right| \leq f(x).$$

Since $f \in L^p(\mathbb{R})$, by the LDCT,

$$\lim_{n\to\infty}\frac{1}{n}\int_0^n xf(x)\,dx = \lim_{n\to\infty}\int_{\mathbb{R}}\frac{x}{n}\chi_{[0,n]}f(x)\,dx = \int_{\mathbb{R}} f(x)\lim_{n\to\infty}\frac{x}{n}\chi_{[0,n]}\,dx = \int_{\mathbb{R}} f(x)\cdot 0\,dx = 0.$$

2. Solution. (**a**) Let A be divided up into three overlapping sections as shown at right, call them A_1, A_2, A_2.

Picture here

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[23] Remark: one can repeat this argument with any growth condition of the form $|f(z)| \leq |z|^p$ and Cauchy's formula for a suitable choice of nth derivative and show that f has to be a polynomial of degree depending on p.

On each A_i, since it is simply connected, there exists an analytic function f_i such that

$$\operatorname{Re} f_i = u \qquad \text{on } A_i.$$

This gives us three (probably) different functions. However they all differ by an imaginary constant, for example on $A_1 \cap A_2$,

$$\operatorname{Re}(f_1 - f_2) = u - u = 0.$$

By Cauchy-Riemann, any holomorphic function with constant real part has constant imaginary part.[24] Applying this to the holomorphic function $f_1 - f_2$ on $A_1 \cap A_2$, we can write $f_1 = f_2 + iK_2$. Similarly

MISSING

3. Solution. (**a**) Let $y = nx$, so $dy = n\,dx$ and

$$\int_{|x|\geq\delta} nk(nx)\,dx = \int_{|y|\geq n\delta} k(y)\,dy.$$

By the MCT, since $k \geq 0$,

$$\lim_{m\longrightarrow\infty} \int_{-m}^{m} k(y)\,dy = \int_{\mathbb{R}} k(y)\,dy.$$

As $n \to \infty$, $n\delta \to \infty$, and thus by additivity

$$\lim_{n\longrightarrow\infty} \int_{|y|\geq n\delta} k(y)\,dy = \lim_{n\longrightarrow\infty} \left\{ \int_{\mathbb{R}} k(y)\,dy - \int_{-n\delta}^{n\delta} k(y)\,dy \right\} = 1 - 1 = 0.$$

(**b**) Again making the change of variables $y = nx$, the integrand becomes $k(y)g(y/n)\,dy$. Since g is bounded, the integrand is dominated by the integrable function $k(y)$, so by the LDCT

$$\lim_{n\longrightarrow\infty} \int_{\mathbb{R}} k(y)g(y/n)\,dy = \int_{\mathbb{R}} \lim_{n\longrightarrow\infty} k(y)g(y/n)\,dy = \int_{\mathbb{R}} k(y)g(0)\,dy = g(0),$$

where the penultimate step follows from the fact that g is continuous.[25]

5. Solution. (**a**) The function $x \mapsto f(x,y)$ is decreasing by (i) and has continuous partial by (iii), $f_1(x,y) \leq 0$, so $-f_1(x,y)$ is a nonnegative function. Thus by Tonelli (two times)

$$\int_x^\infty \int_0^\infty -f_1(t,y)\,dy\,dt = \iint_{\{(t,y)\ dt t>x, y>0\}} -f_1(t,y)\,dt \wedge dy = \int_0^\infty \int_x^\infty -f_1(t,y)\,dt\,dy.$$

Since $x \mapsto f(x,y)$ is $\mathbb{C}[1]$, by the (freshman calculus) FTC, for all $b \geq x$,

$$\int_{t=x}^{b} -f_1(t,y)\,dt = \int_{t=b}^{x} \left[\frac{\partial}{\partial t} f(t,y) \right] dt = f(x,y) - f(b,y).$$

---

[24] If $f(z) = u(x,y) + iv(x,y)$ then $u_x = u_y = 0 \implies v_y = -v_x = 0$

[25] Remark: this is probably not the proof that was intended, since part (a) suggests that the δ-ε definition of continuity is supposed to be used for part (b). As a modification to part (b), try solving the following using part (a): If $g \in L^2(\mathbb{R})$ and continuous on $\mathbb{R}$, but not necessarily bounded, and k is also square integrable, show the same result holds.

By (i), as $b \to \infty$, $f(b,y) \to 0$. We obtain

$$-\int_x^\infty \int_0^\infty f_1(t,y)\,dy\,dt = \int_0^\infty f(x,y) - \lim_{b\to\infty} f(b,y)\,dy = \int_0^\infty f(x,y)\,dy = F(x).$$

(**b**) By the Lebesgue lemma, if $g(t) \in L^1(a,b)$, then $G(x) = \int_a^x g(t)\,dt$ is differentiable almost everywhere and $G'(x) = g(x)$. Let $g(t) = \int_0^\infty -f_1(t,y)\,dy$, which must exist a.e. by part (a). Also by part (a), for all $x > 0$, g is integrable with respect to t on the interval (x,∞). So picking any $a \in (x,\infty)$, if we let $G(x) = \int_x^a g(t)\,dt$, $G'(x) = -g(x)$. Since $\int_a^\infty g(t)\,dt$ is constant with respect to x, its derivative is 0. So we get,

$$F'(x) = \frac{d}{dx}\int_x^\infty g(t)\,dt = -g(x) = \int_0^\infty f_1(t,y)\,dy \qquad \text{for almost all } x > 0.$$

13. Solutions for August 2006

3. Solution. Since $(x^p + \frac{1}{x^p})f \in L^2(0,\infty)$, $(x^p + \frac{1}{x^p})^2 f^2 = x^{2p}f^2 + 2f^2 + x^{-2p}f^2 \in L^1(0,\infty)$. Since this is the sum of three nonnegative functions, it follows that each one is integrable on the same domain. So

$$\begin{aligned} x^{2p}f^2 \in L^1(0,\infty) &\Rightarrow x^p f \in L^2(0,\infty) \\ x^{-2p}f^2 \in L^1(0,\infty) &\Rightarrow x^{-p} f \in L^2(0,\infty). \end{aligned}$$

Now we break up the integral over f into the sets $(0,1]$ and $(1,\infty)$ and compute each separately. Over $(0,1]$, we have that $x \mapsto x^p$ is square integrable, so by Hölders,

$$\int_0^1 f(x)\,dx = \int_0^1 x^p x^{-p} f(x)\,dx \le \left(\int_0^1 x^{2p}\,dx\right)^{1/2} \left(\int_0^1 x^{-2p} f^2(x)\,dx\right)^{1/2}$$

... UNFINISHED?

14. Solutions for January 2007

1. Solution. Let $g_n(x) = \frac{x^n}{1+x^n}$. We write

$$\lim_{n\to\infty}\int_0^\infty g_n f = \lim_{n\to\infty}\left\{\int_0^1 g_n f + \int_1^\infty g_n f\right\}$$

and evaluate each integral separately.

Case: $x \in (0,1)$. Then $0 < x^n < 1$, so $g_n(x) < 1$. Thus $g_n f$ is dominated by the integrable function f, so by the LDCT and the fact that $x^n \xrightarrow{n\to\infty} 0$,

$$\lim_{n\to\infty}\int_0^1 g_n(x)f(x) = \int_0^1 \lim_{n\to\infty}\frac{x^n}{1+x^n}f(x) = \int_0^1 0\cdot f(x) = 0.$$

Case: $x \in (1,\infty)$. In this case $x^n > x^{n+1}$, so

$$\frac{1}{g_n(x)} = \frac{1+x^n}{x^n} = \frac{1}{x^n} + 1 > \frac{1}{x^{n+1}} + 1 = \frac{1}{g_n(x)}$$

thus g_n is monotonically increasing with respect to n. Further as $n \to \infty$, $x^n \to \infty$, so $g_n(x) \to 1$. By the MCT,

$$\lim_{n \longrightarrow \infty} \int_1^\infty g_n(x) f(x) = \int_1^\infty \lim_{n \longrightarrow \infty} g_n(x) f(x) = \int_1^\infty f(x).$$

2. Solution. Let ζ be a primitive 8th root of -1. Since $z^8 + 1 = \prod(z - \zeta^j)$, we are looking for a partial fraction decomposition of the form,

$$f(z) = \frac{a_1}{z - \zeta} + \frac{a_2}{z - \zeta^2} + \cdots + \frac{a_7}{z - \zeta^7}$$

for some constants $a_1, \ldots, a_7$.

Now let γ_j be the small circle centered at ζ_j shown at right, so γ_j contains no other power of ζ. Then $f(z)$ has a simple pole at ζ.

$$\left(\frac{1}{f}\right)'(z) = \frac{z^7(8z^7) - (z^8+1)(7z^6)}{(z^7)^2},$$

since $(\zeta^j)^8 = -1$,

$$\left(\frac{1}{f}\right)'(\zeta^j) = \frac{8\zeta^{14j} - (0)(7\zeta^{6j})}{\zeta^{14j}} = 8.$$

Now f has a simple pole at ζ^j, so,[26]

$$\mathrm{Res}_{\zeta^j} \frac{1}{(1/f)} = \frac{1}{(1/f)'(\zeta^j)} = \frac{1}{8}.$$

Therefore by the residue theorem,

$$\int_{\gamma_j} f(z) = 2\pi i (\mathrm{Res}_{\zeta^j}) = \frac{\pi i}{4}.$$

For $i \neq j$, $\frac{P_i(z)}{z - \zeta^i}$ is analytic inside γ^j, so,

$$\frac{\pi i}{4} = \int_{\gamma_j} f(z) = \sum_i \int_{\gamma_j} \frac{a_i}{z - \zeta^i} = \int_{\gamma_j} \frac{a_j}{z - \zeta^i} = a_j$$

giving our partial fraction decomposition.

3. Solution. Let

$$p' = \frac{p+q}{q}, \qquad q' = \frac{p+q}{p} \qquad \text{and,} \qquad r' = \frac{pq}{p+q}$$

Then we have the equalities,

$$\frac{1}{p'} + \frac{1}{q'} = 1, \qquad \frac{1}{r} + \frac{1}{r'} = 1, \qquad p'r' = p \qquad \text{and,} \qquad q'r' = q$$

Notice that $(|f|^{r'})^{p'} = |f|^{r'p'} = |f|^p$, so $|f|^{r'} \in L^{p'}[0,1]$. Similarly, $|g|^{q'} \in L^{q'}[0,1]$. By Hölders inequality applied to the conjugate pair p', q',

$$\begin{aligned} \left(\int_0^1 |f|^{r'} |g|^{r'}\right)^{1/r'} &\leq \left(\int_0^1 |f|^{r'p'}\right)^{1/p'r'} \left(\int_0^1 |g|^{r'q'}\right)^{1/q'r'} \\ &= \left(\int_0^1 |f|^p\right)^{1/p} \left(\int_0^1 |g|^q\right)^{1/p} < \infty \end{aligned}$$

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[26] do this a better way without using this dumb formula. I hate all these formulas... they're impossible to remember

so $fg \in L^{r'}[0,1]$, and $\|fg\|_{r'} \leq \|f\|_p\|g\|_q$. So by Hölders inequality applied to the conjugate pair r, r', $(fg)h$ is integrable and,

$$\|fgh\|_1 \leq \|fg\|_{r'}\|h\|_r \leq \|f\|_p\|g\|_q\|h\|_r.$$

4. Solution. (**a**) We rewrite the indexing and coefficients so that g looks like a normal power series,

$$g(z) = \sum_{n=1}^{\infty} b_n z^n$$

where $b_n = 0$ unless $n = 2^m$ for some m, in which case $b_n = \frac{1}{m!}$. Then the nonzero terms of $|b_n|^{1/n}$ are a decreasing positive sequence and therefore have a limit, which will be the $\overline{\lim}$ of $|b_n|^{1/n}$. By continuity,

$$\overline{\lim_{n\longrightarrow\infty}} |b_n|^{1/n} = \lim_{m\longrightarrow\infty} |\frac{1}{m!}|^{1/2^m} = \exp\left(\lim_{m\longrightarrow\infty} \frac{1}{2^m}\ln(\frac{1}{m!})\right) = \exp\left(\lim_{m\longrightarrow\infty} \frac{-\ln(m!)}{2^m}\right)$$

For all $m > 1$, $m! > m$, so since ln is increasing, $\ln(m!) \geq \ln(m)$, so finally by L'Hopitol,

$$\lim_{m\longrightarrow\infty} \frac{-\ln(m!)}{2^m} \leq \lim_{m\longrightarrow\infty} \frac{-\ln(m)}{2^m} = \lim_{m\longrightarrow\infty} \frac{-1/m}{\ln 2 \cdot 2^m} = 0.$$

Thus $\overline{\lim}_{n\longrightarrow\infty} |b_n|^{1/n} = \exp(0) = 1$, so the radius of convergence of the power series is 1.

(**b**) For $m \geq 2^n$, $(e^{2\pi ik/2^n}z)^m = z^m$, so the power series of g is

$$\begin{aligned} g(e^{2\pi ik/2^n}z) &= \sum_{j=1}^{m-1} \frac{(e^{2\pi ik/2^n}z)^j}{j!} + \sum_{j=m}^{\infty} \frac{z^j}{j!} \\ &= \left(\sum_{j=1}^{m-1} \frac{(e^{2\pi ik/2^n}z)^j}{j!} - \sum_{j=1}^{m-1} \frac{z^j}{j!}\right) + g(z) \overset{\text{call}}{=} p(z) + g(z) \end{aligned}$$

(**c**) Suppose g has an extension from $\mathbb{D}$, so there exists a path γ leaving $\mathbb{D}$ which has a neighborhood where g is analytic. So γ crosses $\partial\mathbb{D}$, so there exists $\zeta \in \partial\mathbb{D}$, with disk of radius δ such that g is analytic on the ball of radius δ about ζ. Pick any $z \in \partial\mathbb{D}$. By the density of $\{e^{2\pi ik/2^n}\}$ in the unit circle, there are a k, n such that $|e^{2\pi ik/2^n}\zeta - z| < \delta/2$. But by part b, on the ball of radius δ about $e^{2\pi ik/2^n}\zeta$, $g(z) = g(e^{2\pi ik/2^n}z) - p(z)$. Since p is entire, and $e^{2\pi ik/2^n}z$ lands inside a neighborhood where we assumed g was analytic, g is analytic at z.

(**d**) If g an analytic extension, g would be analytic on the entire unit circle by part c, and would therefore have radius of convergence $r > 1$, contradicting part a.

15. Solutions for August 2007

4. Solution. This is a standard problem from Complex Dynamics; of course if you haven't seen complex dynamics it looks strange. The three steps of this proof are three very important techniques to memorize.

The iterates form a normal family on the unit disk. Write $g = \varphi \circ \psi$, where

$$\varphi(z) = z^2, \qquad , \psi(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}$$

and $\alpha = -1/2$. Then it is a standard result that ψ an analytic isomorphism of the unit disk, while φ maps the unit disk into itself. Thus g maps the unit disk into itself so its iterates are uniformly bounded. By Montel's theorem, $\{g^n\}$ forms a normal family on the unit disk.

Near 1, the iterates converge pointwise to 1. In the language of complex dynamics, 1 is an attracting fixed point (z s.t. $g(z) = z$ and $|g'(z)| < 1$) and this is generally true of all attracting fixed points. Here's why. First it's easy to check that g is analytic near 1 and that 1 is in fact a fixed point. A direct computation shows

$$|g'(1)| = \frac{2}{3}$$

Then near 1, $g(z) = 1 + \frac{2}{3}(z-1) + O((z-1)^2)$, and now notice that $g^n(z) = 1 + (\frac{2}{3})^n(z - 1) + O((z-1)^2)$. Even closer to 1, this behaves like $1 + (\frac{2}{3})^n(z-1)$, which since $|\frac{2}{3}| < 1$, converges to 1 as $n \to \infty$.

If a normal family converges pointwise to a constant function, then it converges normally.[27]

We claim that $g^n \to 1$ normally on the whole unit disk. Suppose not. Then some subsequence stays away from 1; formally, there is a compact set K, $\varepsilon > 0$, and subsequence $\{g^{n_k}\}$ such that for all k,

$$\sup_{z \in K} |g^{n_k}(z) - 1| > \varepsilon \qquad (*)$$

A subset of a normal family is normal, so there is a further subsequence $\{g^{n_{k_j}}\}$ converging normally on the unit disk to some function, say

$$G(z) = \lim_{n \longrightarrow \infty} g^{n_{k_j}}(z)$$

The normal limit of analytic functions is analytic (Cauchy's formula) so G is analytic near 1.[28] But we know $g^{n_{k_j}}(z) \to 1$ on a possibly small neighborhood of 1, thus G is identically 1 near 1, and since it's analytic,

$$G(z) = 1 \qquad \text{for all } |z| < 1$$

But now we have a contradiction since we can't have both $g^{n_{k_j}}(z) \to 1$ and (*), that is since $g^{n_{k_j}}(z) \to 1$ there exists an index J such that for all $j > J$,

$$\lim_{z \in K} |g^{n_{k_j}}(z) - 1| < 1.$$

Remark: there's another approach to this problem that combines the first two steps. What you can do is show that for all real numbers near 1, the iterates of g converge

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[27] This fact is related to a result known as Vitali's theorem, but you should know how to prove it. We say "converges normally" to mean uniformly on compact subsets. See the problem is that if you're a normal family, you only know that a subsequence converges normally, so why does the whole family converge normally? This comes up a lot.

[28] technically all we know is that it's analytic on the intersection of some neighborhood of 1 and the unit disk, but that doesn't matter.

pointwise to 1. For this you just need standard real analysis. Now we just need to slightly modify the last step... instead of convergence to a function that's identically 1 on a whole neighborhood, it's only identically 1 on a sliver of the real line near 1. But recall that an analytic function is uniquely defined by its values on a set that accumulates; certainly this sliver of the real line accumulates to 1, and we still get that the function $G = \lim g^{n_{k_j}}$ has to be identically 1. Proceed the same as before.

5. Solution. By additivity,

$$\int_{\mathbb{R}} |f| = \int_{|x|\leq 1} |f| + \int_{|x|>1} |f|.$$

Now we evaluate each integral separately.

($\int_{|x|\leq 1} |f|$) In this case, the constant function $\chi_{(-1,1)} \in L^2(-1,1)$, and since $f \in L^2(\mathbb{R})$, by the Cauchy-Schwarz inequality,

$$\int_{|x|\leq 1} |f| \leq \left(\int_{-1}^{1} 1^2\right)^{1/2} \left(\int_{-1}^{1} |f|^2\right)^{1/2} = \sqrt{2}\left(\int_{-1}^{1} |f|^2\right)^{1/2} \leq \sqrt{2}\, \|f\|_2$$

where $\|f\|_2$ is the norm of f in $L^2(\mathbb{R})$.

($\int_{|x|>1} |f|$) By basic calculus, $1/x^2$ is integrable on the set $|x| > 1$, so $1/x \in L^2(\mathbb{R} - [-1,1])$, and satisfies

$$\int_{|x|>1} \frac{1}{x^2} = 2\int_{1}^{\infty} \frac{1}{x^2} = 2\left[\lim_{b\longrightarrow\infty} -\frac{1}{x}\bigg|_1^b\right] = 2.$$

So again by Cauchy-Schwarz,

$$\int_{|x|>1} |f| \leq \left(\int_{|x|>1} \frac{1}{x^2}\right)^{1/2} \left(\int_{|x|>1} |fx|^2\right)^{1/2} = \sqrt{2}\, \|fx\|_2.$$