# Actuarial Mathematics and Life-Table Statistics

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## Chapter 4

## Expected Present Values of Insurance Contracts

We are now ready to draw together the main strands of the development so far: (i) expectations of discrete and continuous random variables defined as functions of a life-table waiting time T until death, and (ii) discounting of future payment (streams) based on interest-rate assumptions. The approach is first to define the contractual terms of and discuss relations between the major sorts of insurance, endowment and life annuity contracts, and next to use interest theory to define the present value of the contractual payment stream by the insurer as a nonrandom function of the random individual lifetime T. In each case, this leads to a formula for the expected present value of the payout by the insurer, an amount called the **net single premium** or **net single risk premium** of the contract because it is the single cash payment by the insured at the beginning of the insurance period which would exactly compensate for the average of the future payments which the insurer will have to make.

The details of the further mathematical discussion fall into two parts: first, the specification of formulas in terms of cohort life-table quantities for net single premiums of insurances and annuities which pay only at whole-year intervals; and second, the application of the various survival assumptions concerning interpolation between whole years of age, to obtain the corresponding formulas for insurances and annuities which have m payment times per year. We close this Chapter with a discussion of instantaneous-payment insurance,

continuous-payment annuity, and mean-residual-life formulas, all of which involve continuous-time expectation integrals. We also relate these expectations with their m-payment-per-year discrete analogues, and compare the corresponding integral and summation formulas.

Similar and parallel discussions can be found in the *Life Contingencies* book of Jordan (1967) and the *Actuarial Mathematics* book of Bowers et al. (1986). The approach here differs in unifying concepts by discussing together all of the different contracts, first in the whole-year case, next under interpolation assumptions in the *m*-times-per-year case, and finally in the instantaneous case.

#### 4.1 Expected Present Values of Payments

Throughout the Chapter and from now on, it is helpful to distinguish notationally the expectations relating to present values of life insurances and annuities for a life aged x. Instead of the notation  $E(g(T) | T \ge x)$  for expectations of functions of life-length random variables, we define

$$\mathcal{E}_x (g(T)) = E(g(T) \mid T \ge x)$$

The expectations formulas can then be written in terms of the residual-lifetime variable S = T - x (or the change-of-variable s = t - x) as follows:

$$\mathcal{E}_{x}\left(g(T)\right) = \int_{x}^{\infty} g(t) \frac{f(t)}{S(x)} dt = \int_{x}^{\infty} g(t) \frac{\partial}{\partial t} \left(-\frac{S(t)}{S(x)}\right) dt$$
$$= \int_{0}^{\infty} g(s+x) \frac{\partial}{\partial s} \left(-sp_{x}\right) ds = \int_{0}^{\infty} g(s+x) \mu(s+x) sp_{x} ds$$

#### 4.1.1 Types of Insurance & Life Annuity Contracts

There are three types of contracts to consider: insurance, life annuities, and endowments. More complicated kinds of contracts — which we do not discuss in detail — can be obtained by combining (superposing or subtracting) these in various ways. A further possibility, which we address in Chapter 10, is

to restrict payments to some further contingency (e.g., death-benefits only under specified cause-of-death).

In what follows, we adopt several uniform notations and assumptions. Let x denote the initial age of the holder of the insurance, life annuity, or endowment contract, and assume for convenience that the contract is initiated on the holder's birthday. Fix a nonrandom effective (i.e., APR) interest rate i, and retain the notation  $v = (1+i)^{-1}$ , together with the other notations previously discussed for annuities of nonrandom duration. Next, denote by m the number of payment-periods per year, all times being measured from the date of policy initiation. Thus, for given m, insurance will pay off at the end of the fraction 1/m of a year during which death occurs, and life-annuities pay regularly m times per year until the annuitant dies. The term or duration n of the contract will always be assumed to be an integer multiple of 1/m. Note that policy durations are all measured from policy initiation, and therefore are exactly x smaller than the exact age of the policyholder at termination.

The random exact age at which the policyholder dies is denoted by T, and all of the contracts under discussion have the property that T is the only random variable upon which either the amount or time of payment can depend. We assume further that the payment amount depends on the time T of death only through the attained age  $T_m$  measured in multiples of 1/m year. As before, the survival function of T is denoted S(t), and the density either f(t). The probabilities of the various possible occurrences under the policy are therefore calculated using the conditional probability distribution of T given that  $T \geq x$ , which has density f(t)/S(x) at all times  $t \geq x$ . Define from the random variable T the related discrete random variable

$$T_m = \frac{[Tm]}{m}$$
 = age at beginning of  $\frac{1}{m}th$  of year of death

which for integer initial age x is equal to x + k/m whenever  $x + k/m \le T < x + (k+1)/m$ . Observe that the probability mass function of this random variable is given by

$$P(T_m = x + \frac{k}{m} \mid T \ge x) = P(\frac{k}{m} \le T - x < \frac{k+1}{m} \mid T \ge x)$$

$$= \frac{1}{S(x)} \left[ S(x + \frac{k}{m}) - S(x + \frac{k+1}{m}) \right] = {}_{k/m} p_x - {}_{(k+1)/m} p_x$$

$$= P(T \ge x + \frac{k}{m} \mid T \ge x) \cdot P(T < x + \frac{k+1}{m} \mid T \ge x + \frac{k}{m})$$
 (4.1)

$$= {}_{k/m}p_x \cdot {}_{1/m}q_{x+k/m}$$

As has been mentioned previously, a key issue in understanding the special nature of life insurances and annuities with multiple payment periods is to understand how to calculate or interpolate these probabilities from the probabilities  $jp_y$  (for integers j, y) which can be deduced or estimated from life-tables.

An **Insurance** contract is an agreement to pay a face amount — perhaps modified by a specified function of the time until death — if the insured, a life aged x, dies at any time during a specified period, the term of the policy, with payment to be made at the end of the 1/m year within which the death occurs. Usually the payment will simply be the face amount F(0), but for example in decreasing term policies the payment will be  $F(0) \cdot (1 - \frac{k-1}{nm})$  if death occurs within the  $k^{\text{th}}$  successive fraction 1/m year of the policy, where n is the term. (The insurance is said to be a whole-life policy if  $n = \infty$ , and a term insurance otherwise.) The general form of this contract, for a specified term  $n \leq \infty$ , payment-amount function  $F(\cdot)$ , and number m of possible payment-periods per year, is to

pay F(T-x) at time  $T_m - x + \frac{1}{m}$  following policy initiation, if death occurs at T between x and x + n.

The present value of the insurance company's payment under the contract is evidently

$$\begin{cases}
F(T-x) v^{T_m-x+1/m} & \text{if } x \leq T < x+n \\
0 & \text{otherwise}
\end{cases}$$
(4.2)

The simplest and most common case of this contract and formula arise when the face-amount F(0) is the constant amount paid whenever a death within the term occurs. Then the payment is F(0), with present value  $F(0) v^{-x+([mT]+1)/m}$ , if  $x \leq T < x + n$ , and both the payment and present value are 0 otherwise. In this case, with  $F(0) \equiv 1$ , the net single premium has the standard notation  $A^{(m)}_{\overline{x};\overline{n}}$ . In the further special case where m = 1,

the superscript m is dropped, and the net single premium is denoted  $A^1_{\overline{x:n}}$ . Similarly, when the insurance is whole-life  $(n = \infty)$ , the subscript n and bracket  $\overline{n}$  are dropped.

A Life Annuity contract is an agreement to pay a scheduled payment to the policyholder at every interval 1/m of a year while the annuitant is alive, up to a maximum number of nm payments. Again the payment amounts are ordinarily constant, but in principle any nonrandom time-dependent schedule of payments F(k/m) can be used, where F(s) is a fixed function and s ranges over multiples of 1/m. In this general setting, the life annuity contract requires the insurer to

pay an amount F(k/m) at each time  $k/m \le T - x$ , up to a maximum of nm payments.

To avoid ambiguity, we adopt the convention that in the finite-term life annuities, either F(0) = 0 or F(n) = 0. As in the case of annuities certain (i.e., the nonrandom annuities discussed within the theory of interest), we refer to life annuities with first payment at time 0 as (life) annuities-due and to those with first payment at time 1/m (and therefore last payment at time n in the case of a finite term n over which the annuitant survives) as (life) annuities-immediate. The present value of the insurance company's payment under the life annuity contract is

$$\sum_{k=0}^{(T_m-x)m} F(k/m) v^{k/m}$$
 (4.3)

Here the situation is definitely simpler in the case where the payment amounts F(k/m) are level or constant, for then the life-annuity-due payment stream becomes an annuity-due certain (the kind discussed previously under the Theory of Interest) as soon as the random variable T is fixed. Indeed, if we replace F(k/m) by 1/m for  $k=0,1,\ldots,nm-1$ , and by 0 for larger indices k, then the present value in equation (4.3) is  $\ddot{\mathbf{a}}_{\frac{min(T_m+1/m,n)}{x:n}}^{(m)}$ , and its expected present value (= net single premium) is denoted  $\ddot{\mathbf{a}}_{\frac{x}{x:n}}^{(m)}$ .

In the case of temporary life annuities-immediate, which have payments commencing at time 1/m and continuing at intervals 1/m either until death or for a total of nm payments, the expected-present value notation

is  $\mathbf{a}_{\overline{x:n}}^{(m)}$ . However, unlike the case of annuities-certain (i.e., nonrandom-duration annuities), one cannot simply multiply the present value of the life annuity-due for fixed T by the discount-factor  $v^{1/m}$  in order to obtain the corresponding present value for the life annuity-immediate with the same term n. The difference arises because the payment streams (for the life annuity-due deferred 1/m year and the life-annuity immediate) end at the same time rather than with the same number of payments when death occurs before time n. The correct conversion-formula is obtained by treating the life annuity-immediate of term n as paying, in all circumstances, a present value of 1/m (equal to the cash payment at policy initiation) less than the life annuity-due with term n+1/m. Taking expectations leads to the formula

$$\mathbf{a}_{\overline{x:n|}}^{(m)} = \ddot{\mathbf{a}}_{\overline{x:n+1/m|}}^{(m)} - 1/m$$
 (4.4)

In both types of life annuities, the superscripts  $^{(m)}$  are dropped from the net single premium notations when m=1, and the subscript n is dropped when  $n=\infty$ .

The third major type of insurance contract is the **Endowment**, which pays a contractual face amount F(0) at the end of n policy years if the policyholder initially aged x survives to age x + n. This contract is the simplest, since neither the amount nor the time of payment is uncertain. The pure endowment contract commits the insurer to

pay an amount F(0) at time n if  $T \ge x + n$ 

The present value of the pure endowment contract payment is

$$F(0) v^n$$
 if  $T \ge x + n$ , 0 otherwise (4.5)

The net single premium or expected present value for a pure endowment contract with face amount F(0) = 1 is denoted  $A_{\overline{x:n}}^{1}$  or  ${}_{n}E_{x}$  and is evidently equal to

$$A_{\overline{x:n}|}^{1} = {}_{n}E_{x} = v^{n}{}_{n}p_{x} \tag{4.6}$$

The other contract frequently referred to in beginning actuarial texts is the **Endowment Insurance**, which for a life aged x and term n is simply the sum of the pure endowment and the term insurance, both with term n and the same face amount 1. Here the contract calls for the insurer to

pay \$1 at time  $T_m + \frac{1}{m}$  if T < n, and at time n if  $T \ge n$ 

The present value of this contract has the form  $v^n$  on the event  $[T \ge n]$  and the form  $v^{T_m-x+1/m}$  on the complementary event [T < n]. Note that  $T_m + 1/m \le n$  whenever T < n. Thus, in both cases, the present value is given by

$$v^{\min(T_m - x + 1/m, n)} \tag{4.7}$$

The expected present value of the unit endowment insurance is denoted  $A_{\overline{x:n}}^{(m)}$ . Observe (for example in equation (4.10) below) that the notations for the net single premium of the term insurance and of the pure endowment are intended to be mnemonic, respectively denoting the parts of the endowment insurance determined by the expiration of life — and therefore positioning the superscript 1 above the x — and by the expiration of the fixed term, with the superscript 1 in the latter case positioned above the n.

Another example of an insurance contract which does not need separate treatment, because it is built up simply from the contracts already described, is the n-year deferred insurance. This policy pays a constant face amount at the end of the time-interval 1/m of death, but only if death occurs after time n, i.e., after age x+n for a new policyholder aged precisely x. When the face amount is 1, the contractual payout is precisely the difference between the unit whole-life insurance and the n-year unit term insurance, and the formula for the net single premium is

$$A_x^{(m)} - A_{\overline{x};\overline{n}}^{(m)}$$
 (4.8)

Since this insurance pays a benefit only if the insured survives at least n years, it can alternatively be viewed as an endowment with benefit equal to a whole life insurance to the insured after n years (then aged x + n) if the insured lives that long. With this interpretation, the n-year deferred insurance has net single premium  $= {}_{n}E_{x} \cdot A_{x+n}$ . This expected present value must therefore be equal to (4.8), providing the identity:

$$A_x^{(m)} - A_{x=n}^{(m)1} = v^n {}_{n} p_x \cdot A_{x+n}$$
 (4.9)

#### 4.1.2 Formal Relations among Net Single Premiums

In this subsection, we collect a few useful identities connecting the different types of contracts, which hold without regard to particular life-table interpolation assumptions. The first, which we have already seen, is the definition of endowment insurance as the superposition of a constant-face-amount term insurance with a pure endowment of the same face amount and term. In terms of net single premiums, this identity is

$$A_{\overline{x:n}|}^{(m)} = A^{(m)1}_{\overline{x:n}|} + A^{(m)1}_{\overline{x:n}|}$$
(4.10)

The other important identity concerns the relation between expected present values of endowment insurances and life annuities. The great generality of the identity arises from the fact that, for a fixed value of the random lifetime T, the present value of the life annuity-due payout coincides with the annuity-due certain. The unit term-n life annuity-due payout is then given by

$$\ddot{\mathbf{a}}_{\min(T_m - x + 1/m, n)}^{(m)} = \frac{1 - v^{\min(T_m - x + 1/m, n)}}{d^{(m)}}$$

The key idea is that the unit life annuity-due has present value which is a simple linear function of the present value  $v^{\min(T_m-x+1/m,n)}$  of the unit endowment insurance. Taking expectations (over values of the random variable T, conditionally given  $T \geq x$ ) in the present value formula, and substituting  $A_{\overline{x}:n}^{(m)}$  as expectation of (4.7), then yields:

$$\ddot{\mathbf{a}}_{\overline{x:n}|}^{(m)} = \mathcal{E}_x \left( \frac{1 - v^{\min(T_m - x + 1/m, n)}}{d^{(m)}} \right) = \frac{1 - A_{\overline{x:n}|}^{(m)}}{d^{(m)}}$$
(4.11)

where recall that  $\mathcal{E}_x(\cdot)$  denotes the conditional expectation  $E(\cdot | T \ge x)$ . A more common and algebraically equivalent form of the identity (4.11) is

$$d^{(m)} \ddot{\mathbf{a}}_{\overline{x}:\overline{n}|}^{(m)} + A_{\overline{x}:\overline{n}|}^{(m)} = 1 \tag{4.12}$$

To obtain a corresponding identity relating net single premiums for life annuities-immediate to those of endowment insurances, we appeal to the conversion-formula (4.4), yielding

$$\mathbf{a}_{x:n}^{(m)} = \ddot{\mathbf{a}}_{x:n+1/m|}^{(m)} - \frac{1}{m} = \frac{1 - A_{x:n+1/m|}^{(m)}}{d^{(m)}} - \frac{1}{m} = \frac{1}{i^{(m)}} - \frac{1}{d^{(m)}} A_{x:n+1/m|}^{(m)}$$
(4.13)

and

$$d^{(m)} a_{\overline{x:n|}}^{(m)} + A_{\overline{x:n+1/m|}}^{(m)} = \frac{d^{(m)}}{i^{(m)}} = v^{1/m}$$
(4.14)

In these formulas, we have made use of the definition

$$\frac{m}{d^{(m)}} = \left(1 + \frac{i^{(m)}}{m}\right) / \left(\frac{i^{(m)}}{m}\right)$$

leading to the simplifications

$$\frac{m}{d^{(m)}} = \frac{m}{i^{(m)}} + 1$$
 ,  $\frac{i^{(m)}}{d^{(m)}} = 1 + \frac{i^{(m)}}{m} = v^{-1/m}$ 

#### 4.1.3 Formulas for Net Single Premiums

This subsection collects the expectation-formulas for the insurance, annuity, and endowment contracts defined above. Throughout this Section, the same conventions as before are in force (integer x and n, fixed m, i, and conditional survival function  $_tp_x$ ).

First, the expectation of the present value (4.2) of the random term insurance payment (with level face value  $F(0) \equiv 1$ ) is

$$A_{\overline{x:n}|}^{1} = \mathcal{E}_{x} \left( v^{T_{m}-x+1/m} \right) = \sum_{k=0}^{nm-1} v^{(k+1)/m} {}_{k/m} p_{x} {}_{1/m} q_{x+k/m}$$
 (4.15)

The index k in the summation formula given here denotes the multiple of 1/m beginning the interval [k/m, (k+1)/m) within which the policy age T-x at death is to lie. The summation itself is simply the weighted sum, over all indices k such that k/m < n, of the present values  $v^{(k+1)/m}$  to be paid by the insurer in the event that the policy age at death falls in [k/m, (k+1)/m) multiplied by the probability, given in formula (4.1), that this event occurs.

Next, to figure the expected present value of the life annuity-due with term n, note that payments of 1/m occur at all policy ages k/m,  $k = 0, \ldots, nm-1$ , for which  $T-x \ge k/m$ . Therefore, since the present values

of these payments are  $(1/m) v^{k/m}$  and the payment at k/m is made with probability  $_{k/m} p_x$  ,

$$\ddot{\mathbf{a}}_{\overline{x:n}}^{(m)} = \mathcal{E}_x \left( \sum_{k=0}^{nm-1} \frac{1}{m} v^{k/m} I_{[T-x \ge k/m]} \right) = \frac{1}{m} \sum_{k=0}^{nm-1} v^{k/m} {}_{k/m} p_x \qquad (4.16)$$

Finally the pure endowment has present value

$${}_{n}E_{x} = \mathcal{E}_{x} \left( v^{n} I_{[T-x \geq n]} \right) = v^{n} {}_{x} p_{n}$$

$$\tag{4.17}$$

#### **4.1.4** Expected Present Values for m = 1

It is clear that for the general insurance and life annuity payable at whole-year intervals (m=1), with payment amounts determined solely by the whole-year age [T] at death, the net single premiums are given by discrete-random-variable expectation formulas based upon the present values (4.2) and (4.3). Indeed, since the events  $\{[T] \geq x\}$  and  $\{T \geq x\}$  are identical for integers x, the discrete random variable [T] for a life aged x has conditional probabilities given by

$$P([T] = x + k \mid T \ge x) = {}_{k}p_{x} - {}_{k+1}p_{x} = {}_{k}p_{x} \cdot q_{x+k}$$

Therefore the expected present value of the term-n insurance paying F(k) at time k+1 whenever death occurs at age T between x+k and x+k+1 (with k < n) is

$$E\left(v^{[T]-x+1} F([T]-x) I_{[T \le x+n]} \mid T \ge x\right) = \sum_{k=0}^{n-1} F(k) v^{k+1} {}_{k} p_{x} q_{x+k}$$

Here and from now on, for an event B depending on the random lifetime T, the notation  $I_B$  denotes the so-called *indicator random variable* which is equal to 1 whenever T has a value such that the condition B is satisfied and is equal to 0 otherwise. The corresponding life annuity which pays F(k) at each  $k = 0, \ldots, n$  at which the annuitant is alive has expected present value

$$\mathcal{E}_{x}\left(\sum_{k=0}^{\min(n,\,[T]-x)} v^{k} F(k)\right) = \mathcal{E}_{x}\left(\sum_{k=0}^{n} v^{k} F(k) I_{[T \geq x+k]}\right) = \sum_{k=0}^{n} v^{k} F(k) {}_{k} p_{x}$$

In other words, the payment of F(k) at time k is received only if the annuitant is alive at that time and so contributes expected present value equal to  $v^k F(k)_k p_x$ . This makes the annuity equal to the superposition of pure endowments of terms k = 0, 1, 2, ..., n and respective face-amounts F(k).

In the most important special case, where the non-zero face-amounts F(k) are taken as constant, and for convenience are taken equal to 1 for  $k = 0, \ldots, n-1$  and equal to 0 otherwise, we obtain the useful formulas

$$A_{\overline{x:n}}^{1} = \sum_{k=0}^{n-1} v^{k+1} {}_{k} p_{x} q_{x+k}$$

$$(4.18)$$

$$\ddot{\mathbf{a}}_{\overline{x:n}} = \sum_{k=0}^{n-1} v^k{}_k p_x \tag{4.19}$$

$$A_{\overline{x:n|}}^{1} = \mathcal{E}_x \left( v^n I_{[T-x \ge n]} \right) = v^n {}_n p_x \tag{4.20}$$

$$A_{\overline{x:n}} = \sum_{k=0}^{\infty} v^{\min(n,k+1)} {}_k p_x q_{x+k}$$

$$= \sum_{k=0}^{n-1} v^{k+1} \left( {}_{k} p_{x} - {}_{k+1} p_{x} \right) + v^{n} {}_{n} p_{x}$$
 (4.21)

Two further manipulations which will complement this circle of ideas are left as exercises for the interested reader: (i) first, to verify that formula (4.19) gives the same answer as the formula  $\mathcal{E}_x(\ddot{\mathbf{a}}_{x:\min([T]-x+1,\,n]})$ ; and (ii) second, to sum by parts (collecting terms according to like subscripts k of  $_kp_x$  in formula (4.21)) to obtain the equivalent expression

$$1 + \sum_{k=0}^{n-1} (v^{k+1} - v^k)_k p_x = 1 - (1 - v) \sum_{k=0}^{n-1} v^k_k p_x$$

The reader will observe that this final expression together with formula (4.19) gives an alternative proof, for the case m = 1, of the identity (4.12).

Let us work out these formulas analytically in the special case where [T] has the  $Geometric(1-\gamma)$  distribution, i.e., where

$$P([T] = k) = P(k \le T < k + 1) = \gamma^k (1 - \gamma)$$
 for  $k = 0, 1, ...$ 

with  $\gamma$  a fixed constant parameter between 0 and 1. This would be true if the force of mortality  $\mu$  were constant at all ages, i.e., if T were exponentially distributed with parameter  $\mu$ , with  $f(t) = \mu e^{-\mu t}$  for  $t \ge 0$ . In that case,  $P(T \ge k) = e^{-\mu k}$ , and  $\gamma = P(T = k | T \ge k) = 1 - e^{-\mu}$ . Then

$$_{k}p_{x} q_{x+k} = P([T] = x + k \mid T \ge x) = \gamma^{k} (1 - \gamma)$$
 ,  $_{n}p_{x} = \gamma^{n}$ 

so that

$$A_{\overline{x:n}}^{1} = (\gamma v)^{n}$$
 ,  $A_{\overline{x:n}}^{1} = \sum_{k=0}^{n-1} v^{k+1} \gamma^{k} (1-\gamma) = v(1-\gamma) \frac{1-(\gamma v)^{n}}{1-\gamma v}$ 

Thus, for the case of interest rate i = 0.05 and  $\gamma = 0.97$ , corresponding to expected lifetime  $= \gamma/(1-\gamma) = 32.33$  years,

$$A_{\overline{x:20}} = (0.97/1.05)^{20} + \frac{.03}{1.05} \cdot \frac{1 - (.97/1.05)^{20}}{(1 - (.97/1.05))} = .503$$

which can be compared with  $A_x \equiv A_{\overline{x:\infty}}^1 = \frac{.03}{.08} = .375$ .

The formulas (4.18)-(4.21) are benchmarks in the sense that they represent a complete solution to the problem of determining net single premiums without the need for interpolation of the life-table survival function between integer ages. However the insurance, life-annuity, and endowment-insurance contracts payable only at whole-year intervals are all slightly impractical as insurance vehicles. In the next chapter, we approach the calculation of net single premiums for the more realistic context of m-period-per-year insurances and life annuities, using only the standard cohort life-table data collected by integer attained ages.

#### 4.2 Continuous-Time Expectations

So far in this Chapter, all of the expectations considered have been associated with the discretized random lifetime variables [T] and  $T_m = [mT]/m$ . However, Insurance and Annuity contracts can also be defined with respectively instantaneous and continuous payments, as follows. First, an instantaneous-payment or continuous insurance with face-value F

is a contract which pays an amount F at the instant of death of the insured. (In practice, this means that when the actual payment is made at some later time, the amount paid is F together with interest compounded from the instant of death.) As a function of the random lifetime T for the insured life initially with exact integer age x, the present value of the amount paid is  $F \cdot v^{T-x}$  for a whole-life insurance and  $F \cdot v^{T-x} \cdot I_{[T < x+n]}$  for an n-year term insurance. The expected present values or net single premiums on a life aged x are respectively denoted  $\overline{A}_x$  for a whole-life contract and  $\overline{A}_{\overline{x:n}}^1$  for an n-year temporary insurance. The **continuous life annuity** is a contract which provides continuous payments at rate 1 per unit time for duration equal to the smaller of the remaining lifetime of the annuitant or the term of n years. Here the present value of the contractual payments, as a function of the exact age T at death for an annuitant initially of exact integer age x, is  $\overline{a}_{\overline{\min(T-x,n)}}$  where n is the (possibly infinite) duration of the life annuity. Recall that

$$\overline{a}_{\overline{K}|} = \int_0^\infty v^t I_{[t \le K]} dt = \int_0^K v^t dt = (1 - v^K)/\delta$$

is the present value of a continuous payment stream of 1 per unit time of duration K units, where  $v=(1+i)^{-1}$  and  $\delta=\ln(1+i)$ .

The objective of this section is to develop and interpret formulas for these continuous-time net single premiums, along with one further quantity which has been defined as a continuous-time expectation of the lifetime variable T, namely the **mean residual life** (also called **complete life expectancy**)  $\mathring{e}_x = \mathcal{E}_x(T-x)$  for a life aged x. The underlying general conditional expectation formula (1.3) was already derived in Chapter 1, and we reproduce it here in the form

$$\mathcal{E}_{x}\{g(T)\} = \frac{1}{S(x)} \int_{x}^{\infty} g(y) f(y) dy = \int_{0}^{\infty} g(x+t) \mu(x+t) p_{x} dt \quad (4.22)$$

We apply this formula directly for the three choices

$$g(y) = y - x$$
 ,  $v^{y-x}$  , or  $v^{y-x} \cdot I_{[y-x< n]}$ 

which respectively have the conditional  $\mathcal{E}_x(\cdot)$  expectations

$$\stackrel{\circ}{\mathrm{e}}_x$$
 ,  $\overline{\mathrm{A}}_x$  ,  $\overline{\mathrm{A}}_{\overline{x}:\overline{n}}^1$ 

For easy reference, the integral formulas for these three cases are:

$$\stackrel{\circ}{\mathbf{e}}_{x} = \mathcal{E}_{x}(T-x) = \int_{0}^{\infty} t \, \mu(x+t) \,_{t} p_{x} \, dt$$
 (4.23)

$$\overline{A}_x = \mathcal{E}_x(v^{T-x}) = \int_0^\infty v^t \, \mu(x+t) \, _t p_x \, dt$$
 (4.24)

$$\overline{A}_{\overline{x:n}|}^{1} = E_{x} \left( v^{T-x} I_{[T-x \leq n]} \right) = \int_{0}^{n} v^{t} \mu(x+t) {}_{t} p_{x} dt \qquad (4.25)$$

Next, we obtain two additional formulas, for continuous life annuities-due

$$\overline{\mathbf{a}}_x$$
 and  $\overline{\mathbf{a}}_{\overline{x:n}}$ 

which correspond to  $\mathcal{E}_x\{g(T)\}$  for the two choices

$$g(t) = \int_0^\infty v^t I_{[t \le y - x]} dt \quad \text{or} \quad \int_0^n v^t I_{[t \le y - x]} dt$$

After switching the order of the integrals and the conditional expectations, and evaluating the conditional expectation of an indicator as a conditional probability, in the form

$$\mathcal{E}_x\left(I_{[t \le T - x]}\right) = P(T \ge x + t \mid T \ge x) = {}_t p_x$$

the resulting two equations become

$$\overline{\mathbf{a}}_x = \mathcal{E}_x \left( \int_0^\infty v^t I_{[t \le T - x]} dt \right) = \int_0^\infty v^t {}_t p_x dt \tag{4.26}$$

$$\overline{\mathbf{a}}_{\overline{x:n}} = \mathcal{E}_x \left( \int_0^n v^t I_{[t \le T - x]} dt \right) = \int_0^n v^t {}_t p_x dt \qquad (4.27)$$

As might be expected, the continuous insurance and annuity contracts have a close relationship to the corresponding contracts with m payment periods per year for large m. Indeed, it is easy to see that the term insurance net single premiums

$$A^{(m)}\frac{1}{|x-y|} = \mathcal{E}_x \left( v^{T_m - x + 1/m} \right)$$

approach the continuous insurance value (4.24) as a limit when  $m \to \infty$ . A simple proof can be given because the payments at the end of the fraction 1/m of year of death are at most 1/m years later than the continuous-insurance payment at the instant of death, so that the following obvious inequalities hold:

$$\overline{A}_{\overline{x:n}|}^1 \le A^{(m)} \frac{1}{\overline{x:n}|} \le v^{1/m} \overline{A}_{\overline{x:n}|}^1$$
 (4.28)

Since the right-hand term in the inequality (4.28) obviously converges for large m to the leftmost term, the middle term which is sandwiched in between must converge to the same limit (4.25).

For the continuous annuity, (4.27) can be obtained as a limit of formulas (4.16) using Riemann sums, as the number m of payments per year goes to infinity, i.e.,

$$\overline{a}_{\overline{x:n}} = \lim_{m \to \infty} \ddot{a}_{\overline{x:n}}^{(m)} = \lim_{m \to \infty} \sum_{k=0}^{nm-1} \frac{1}{m} v^{k/m}{}_{k/m} p_x = \int_0^n v^t{}_t p_x \, ds$$

The final formula coincides with (4.27), according with the intuition that the limit as  $m \to \infty$  of the payment-stream which pays 1/m at intervals of time 1/m between 0 and  $T_m - x$  inclusive is the continuous payment-stream which pays 1 per unit time throughout the policy-age interval [0, T - x).

Each of the expressions in formulas (4.23), (4.24), and (4.27) can be contrasted with a related approximate expectation for a function of the integer-valued random variable [T] (taking m=1). First, alternative general formulas are developed for the integrals by breaking the formulas down into sums of integrals over integer-endpoint intervals and substituting the definition  $_k p_x/S(x+k) = 1/S(x)$ :

$$\mathcal{E}_{x}(g(T)) = \sum_{k=0}^{\infty} \int_{x+k}^{x+k+1} g(y) \frac{f(y)}{S(x)} dy \quad \text{changing to } z = y - x - k$$

$$= \sum_{k=0}^{\infty} {}_{k} p_{x} \int_{0}^{1} g(x+k+z) \frac{f(x+k+z)}{S(x+k)} dz \qquad (4.29)$$

Substituting into (4.29) the special function g(y) = y - x, leads to

$$\mathring{\mathbf{e}}_{x} = \sum_{k=0}^{\infty} {}_{k} p_{x} \left\{ k \frac{S(x+k) - S(x+k+1)}{S(x+k)} + \int_{0}^{1} z \frac{f(x+k+z)}{S(x+k)} dz \right\}$$
(4.30)

Either of two assumptions between integer ages can be applied to simplify the integrals: (b) (Constant force of mortality)  $\mu(y) = \mu(x+k)$  for  $x+k \leq y < x+k+1$ , in which case  $1-q_{x+k} = \exp(-\mu(x+k))$ .

In case (a), the last integral in (4.30) becomes

$$\int_0^1 z \frac{f(x+k+z)}{S(x+k)} dz = \int_0^1 z \frac{S(x+k) - S(x+k+1)}{S(x+k)} dz = \frac{1}{2} q_{x+k}$$

and in case (b), we obtain within (4.30)

$$\int_0^1 z \, \frac{f(x+k+z)}{S(x+k)} \, dz \; = \; \int_0^1 z \, \mu(x+k) \, e^{-z \, \mu(x+k)} \, dz$$

which in turn is equal (after integration by parts) to

$$-e^{-\mu(x+k)} + \frac{1 - e^{-\mu(x+k)}}{\mu(x+k)} \approx \frac{1}{2}\mu(x+k) \approx \frac{1}{2}q_{x+k}$$

where the last approximate equalities hold if the death rates are small. It follows, exactly in the case (a) where failures are uniformly distributed within integer-age intervals or approximately in case (b) when death rates are small, that

$$\stackrel{\circ}{\mathbf{e}}_{x} = \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right)_{k} p_{x} q_{x+k} = \sum_{k=0}^{\infty} k_{k} p_{x} q_{x+k} + \frac{1}{2}$$
 (4.31)

The final summation in (4.31), called the **curtate life expectancy** 

$$e_x = \sum_{k=0}^{\infty} k_k p_x q_{x+k}$$
 (4.32)

has an exact interpretation as the expected number of whole years of life remaining to a life aged x. The behavior of and comparison between complete and curtate life expectancies is explored numerically in subsection 4.2.1 below.

Return now to the general expression for  $\mathcal{E}_x(g(T))$ , substituting  $g(y) = v^{y-x}$  but restricting attention to case (a):

$$\overline{A}_{\overline{x:n}}^1 = \mathcal{E}\left\{v^{T-x} I_{[T< x+n]}\right\} = \sum_{k=0}^n \int_{x+k}^{x+k+1} v^{y-x} \frac{f(x+k)}{S(x)} dy$$

$$= \sum_{k=0}^{n} \int_{x+k}^{x+k+1} v^{y-x} \frac{S(x+k) - S(x+k+1)}{S(x+k)} {}_{k} p_{x} dy$$

$$= \sum_{k=0}^{n} {}_{k} p_{x} q_{x+k} \int_{0}^{1} v^{k+t} dt = \sum_{k=0}^{n} {}_{k} p_{x} q_{x+k} v^{k+1} \frac{1 - e^{-\delta}}{v \delta}$$

where  $v = 1/(1+i) = e^{-\delta}$ , and  $\delta$  is the force of interest. Since  $1-e^{-\delta} = iv$ , we have found in case (a) that

$$\overline{A}_{\overline{x:n}|}^1 = A_{\overline{x:n}|}^1 \cdot (i/\delta) \tag{4.33}$$

Finally, return to the formula (4.27) under case (b) to find

$$\overline{\mathbf{a}}_{\overline{x:n}} = \sum_{k=0}^{n-1} \int_{k}^{k+1} v^{t} \,_{t} p_{x} \, dt = \sum_{k=0}^{n-1} \int_{k}^{k+1} e^{-(\delta+\mu)t} \, dt$$

$$= \sum_{k=0}^{n-1} \frac{e^{-(\delta+\mu)k} - e^{-(\delta+\mu)(k+1)}}{\delta+\mu} = \sum_{k=0}^{n-1} v^{k} \,_{k} p_{x} \cdot \frac{1 - e^{-(\delta+\mu)}}{\delta+\mu}$$

Thus, in case (b) we have shown

$$\overline{\mathbf{a}}_{\overline{x:n}} = \frac{1 - e^{-(\delta + \mu)n}}{\delta + \mu} = \ddot{\mathbf{a}}_{\overline{x:n}} \cdot \frac{1 - e^{-(\delta + \mu)}}{\delta + \mu} \tag{4.34}$$

In the last two paragraphs, we have obtained formulas (4.33) and (4.34) respectively under cases (a) and (b) relating net single premiums for continuous contracts to those of the corresponding single-payment-per-year contracts. More elaborate relations will be given in the next Chapter between net single premium formulas which do require interpolation-assumptions for probabilities of survival to times between integer ages to formulas for m=1, which do not require such interpolation.

#### 4.2.1 Numerical Calculations of Life Expectancies

Formulas (4.23) or (4.30) and (4.32) above respectively provide the complete and curtate age-specific life expectancies, in terms respectively of survival

densities and life-table data. Formula (4.31) provides the actuarial approximation for complete life expectancy in terms of life-table data, based upon interpolation-assumption (i) (Uniform mortality within year of age). In this Section, we illustrate these formulas using the Illustrative simulated and extrapolated life-table data of Table 1.1.

Life expectancy formulas necessarily involve life table data and/or survival distributions specified out to arbitrarily large ages. While life tables may be based on large cohorts of insured for ages up to the seventies and even eighties, beyond that they will be very sparse and very dependent on the particular small group(s) of aged individuals used in constructing the particular table(s). On the other hand, the fraction of the cohort at moderate ages who will survive past 90, say, is extremely small, so a reasonable extrapolation of a well-established table out to age 80 or so may give sufficiently accurate life-expectancy values at ages not exceeding 80. Life expectancies are in any case forecasts based upon an implicit assumption of future mortality following exactly the same pattern as recent past mortality. Life-expectancy calculations necessarily ignore likely changes in living conditions and medical technology which many who are currently alive will experience. Thus an assertion of great accuracy for a particular method of calculation would be misplaced.

All of the numerical life-expectancy calculations produced for the Figure of this Section are based on the extrapolation (2.9) of the illustrative life table data from Table 1.1. According to that extrapolation, death-rates  $q_x$  for all ages 78 and greater are taken to grow exponentially, with  $log(q_x/q_{78}) = (x - 78) \ln(1.0885)$ . This exponential behavior is approximately but not precisely compatible with a Gompertz-form force-of-mortality function

$$\mu(78+t) = \mu(78) c^t$$

in light of the approximate equality  $\mu(x) \approx q_x$ , an approximation which progressively becomes less valid as the force of mortality gets larger. To see this, note that under a Gompertz survival model,

$$\mu(x) = Bc^x$$
 ,  $q_x = 1 - \exp\left(-Bc^x \frac{c-1}{\ln c}\right)$ 

and with c = 1.0885 in our setting,  $(c-1)/\ln c = 1.0436$ .

Since curtate life expectancy (4.32) relies directly on (extrapolated) lifetable data, its calculation is simplest and most easily interpreted. Figure 4.1 presents, as plotted points, the age-specific curtate life expectancies for integer ages x = 0, 1, ..., 78. Since the complete life expectancy at each age is larger than the curtate by exactly 1/2 under interpolation assumption (a), we calculated for comparison the complete life expectancy at all (real-number) ages, under assumption (b) of piecewise-constant force of mortality within years of age. Under this assumption, by formula (3.11), mortality within year of age (0 < t < 1) is  $_t p_x = (p_x)^t$ . Using formula (4.31) and interpolation assumption (b), the exact formula for complete life expectancy becomes

$$\dot{\mathbf{e}}_x - \mathbf{e}_x = \sum_{k=0}^{\infty} {}_k p_x \left\{ \frac{q_{x+k} + p_{x+k} \ln(p_{x+k})}{-\ln(p_{x+k})} \right\}$$

The complete life expectancies calculated from this formula were found to exceed the curtate life expectancy by amounts ranging from 0.493 at ages 40 and below, down to 0.485 at age 78 and 0.348 at age 99. Thus there is essentially no new information in the calculated complete life expectancies, and they are not plotted.

The aspect of Figure 4.1 which is most startling to the intuition is the large expected numbers of additional birthdays for individuals of advanced ages. Moreover, the large life expectancies shown are comparable to actual US male mortality circa 1959, so would be still larger today.

#### 4.3 Exercise Set 4

- (1). For each of the following three lifetime distributions, find (a) the expected remaining lifetime for an individual aged 20, and (b)  $_{7/12}q_{40}/q_{40}$ .
- (i) Weibull(.00634, 1.2), with  $S(t) = \exp(-0.00634 t^{1.2})$ ,
- (ii)  $Lognormal(\log(50), 0.325^2)$ , with  $S(t) = 1 \Phi((\log(t) \log(50))/0.325)$ ,
- (iii) Piecewise exponential with force of mortality given the constant value  $\mu_t = 0.015$  for  $20 < t \le 50$ , and  $\mu_t = 0.03$  for  $t \ge 50$ . In these integrals, you should be prepared to use integrations by parts, gamma function values, tables of the normal distribution function  $\Phi(x)$ , and/or numerical integrations via calculators or software.

#### Expected number of additional whole years of life, by age

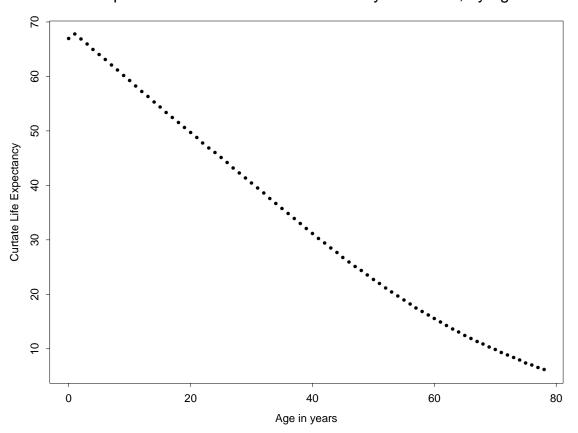


Figure 4.1: Curtate life expectancy  $e_x$  as a function of age, calculated from the simulated illustrative life table data of Table 1.1, with age-specific death-rates  $q_x$  extrapolated as indicated in formula (2.9).

#### 4.3. EXERCISE SET 4

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- (2). (a) Find the expected present value, with respect to the constant effective interest rate r=0.07, of an insurance payment of \$1000 to be made at the instant of death of an individual who has just turned 40 and whose remaining lifetime T-40=S is a continuous random variable with density  $f(s)=0.05\,e^{-0.05\,s}$ , s>0.
- (b) Find the expected present value of the insurance payment in (a) if the insurer is allowed to delay the payment to the end of the year in which the individual dies. Should this answer be larger or smaller than the answer in (a)?
- (3). If the individual in Problem 2 pays a life insurance premium P at the **beginning** of each remaining year of his life (including this one), then what is the expected total present value of all the premiums he pays before his death?
- (4). Suppose that an individual has equal probability of dying within each of the next 40 years, and is certain to die within this time, i.e., his age is x and

$$_{k}p_{x} - _{k+1}p_{x} = 0.025$$
 for  $k = 0, 1, \dots, 39$ 

Assume the fixed interest rate r = 0.06.

- (a) Find the net single whole-life insurance premium  $A_x$  for this individual.
- (b) Find the net single premium for the term and endowment insurances  $A_{\overline{x:20}}^1$  and  $A_{\overline{x:30}}$ .
- (5). Show that the expected whole number of years of remaining life for a life aged x is given by

$$c_x = E([T] - x \mid T \ge x) = \sum_{k=0}^{\omega - x - 1} k_k p_x q_{x+k}$$

and prove that this quantity as a function of integer age x satisfies the recursion equation

$$c_x = p_x (1 + c_{x+1})$$

(6). Show that the expected present value  $b_x$  of an insurance of 1 payable at the beginning of the year of death (or equivalently, payable at the end of

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the year of death along with interest from the beginning of that same year) satisfies the recursion relation (4.35) above.

(7). Prove the identity (4.9) algebraically.

For the next two problems, consider a cohort life-table population for which you know only that  $l_{70} = 10,000, l_{75} = 7000, l_{80} = 3000$ , and  $l_{85} = 0$ , and that the distribution of death-times within 5-year age intervals is uniform.

- (8). Find (a)  $\mathring{e}_{75}$  and (b) the probability of an individual aged 70 in this life-table population dying between ages 72.0 and 78.0.
- (9). Find the probability of an individual aged 72 in this life-table population dying between ages 75.0 and 83.0, if the assumption of uniform death-times within 5-year intervals is replaced by:
- (a) an assumption of constant force of mortality within 5-year ageintervals;
- (b) the Balducci assumption (of linearity of 1/S(t)) within 5-year age intervals.
- (10). Suppose that a population has survival probabilities governed at all ages by the force of mortality

$$\mu_t = \begin{cases} .01 & \text{for } 0 \le t < 1 \\ .002 & \text{for } 1 \le t < 5 \\ .001 & \text{for } 5 \le t < 20 \\ .004 & \text{for } 20 \le t < 40 \\ .0001 \cdot t & \text{for } 40 \le t \end{cases}$$

Then (a) find  $_{30}p_{10}$ , and (b) find  $\mathring{e}_{50}$ .

(11). Suppose that a population has survival probabilities governed at all ages by the force of mortality

$$\mu_t = \begin{cases} .01 & \text{for } 0 \le t < 10 \\ .1 & \text{for } 10 \le t < 30 \\ 3/t & \text{for } 30 \le t \end{cases}$$

Then (a) find  $_{30}p_{20}$  = the probability that an individual aged 20 survives for at least 30 more years, and (b) find  $\mathring{e}_{30}$ .

- (12). Assuming the same force of mortality as in the previous problem, find  $\stackrel{\circ}{e}_{70}$  and  $\overline{A}_{60}$  if i = 0.09.
- (13). The force of mortality for impaired lives is three times the standard force of mortality at all ages. The standard rates  $q_x$  of mortality at ages 95, 96, and 97 are respectively 0.3, 0.4, and 0.5. What is the probability that an impaired life age 95 will live to age 98?
- (14). You are given a survival function  $S(x) = (10-x)^2/100$ ,  $0 \le x \le 10$ .
- (a) Calculate the average number of future years of life for an individual who survives to age 1.
- (b) Calculate the difference between the force of mortality at age 1, and the probability that a life aged 1 dies before age 2.
- (15). An n-year term life insurance policy to a life aged x provides that if the insured dies within the n-year period an annuity-certain of yearly payments of 10 will be paid to the beneficiary, with the first annuity payment made on the policy-anniversary following death, and the last payment made on the  $N^{th}$  policy anniversary. Here  $1 < n \le N$  are fixed integers. If B(x, n, N) denotes the net single premium (= expected present value) for this policy, and if mortality follows the law  $l_x = C(\omega x)/\omega$  for some terminal integer age  $\omega$  and constant C, then find a simplified expression for B(x, n, N) in terms of interest-rate functions,  $\omega$ , and the integers x, n, N. Assume  $x + n \le \omega$ .
- (16). The father of a newborn child purchases an endowment and insurance contract with the following combination of benefits. The child is to receive \$100,000 for college at her  $18^{th}$  birthday if she lives that long and \$500,000 at her  $60^{th}$  birthday if she lives that long, and the father as beneficiary is to receive \$200,000 at the end of the year of the child's death if the child dies before age 18. Find expressions, **both** in actuarial notations and in terms of v = 1/(1+i) and of the survival probabilities  $_k p_0$  for the child, for the net single premium for this contract.

#### 4.4 Worked Examples

Example 1. Toy Life-Table (assuming uniform failures)

Consider the following life-table with only six equally-spaced ages. (That is, assume  $l_6 = 0$ .) Assume that the rate of interest i = .09, so that v = 1/(1+i) = 0.9174 and  $(1-e^{-\delta})/\delta = (1-v)/\delta = 0.9582$ .

X	Age-range	$l_x$	$d_x$	$\mathbf{e}_x$	$\overline{A}_x$
0	0 - 0.99	1000	60	4.2	0.704
1	1 - 1.99	940	80	3.436	0.749
2	2 - 2.99	860	100	2.709	0.795
3	3 - 3.99	760	120	2.0	0.844
4	4 - 4.99	640	140	1.281	0.896
5	5 - 5.99	500	500	0.5	0.958

Using the data in this Table, and interest rate i = .09, we begin by calculating the expected present values for simple contracts for term insurance, annuity, and endowment. First, for a life aged 0, a term insurance with payoff amount \$1000 to age 3 has present value given by formula (4.18) as

$$1000 A_{\overline{0:3}}^{1} = 1000 \left\{ 0.917 \frac{60}{1000} + (0.917)^{2} \frac{80}{1000} + (0.917)^{3} \frac{100}{1000} \right\} = 199.60$$

Second, for a life aged 2, a term annuity-due of \$700 per year up to age 5 has present value computed from (4.19) to be

$$700 \,\ddot{a}_{\overline{2:3}} = 700 \,\left\{ 1 + 0.917 \,\frac{760}{860} + (0.917)^2 \,\frac{640}{860} \right\} = 1705.98$$

For the same life aged 2, the 3-year Endowment for \$700 has present value

$$700 A_{\overline{0:3}|}^{1} = 700 \cdot (0.9174)^{3} \frac{500}{860} = 314.26$$

Thus we can also calculate (for the life aged 2) the present value of the 3-year annuity-immediate of \$700 per year as

$$700 \cdot \left(\ddot{a}_{\overline{2:3}} - 1 + A_{\overline{0:3}}^{-1}\right) = 1705.98 - 700 + 314.26 = 1320.24$$

We next apply and interpret the formulas of Section 4.2, together with the observation that

$$_{j}p_{x} \cdot q_{x+j} = \frac{l_{x+j}}{l_{x}} \cdot \frac{d_{x+j}}{l_{x+j}} = \frac{d_{x+j}}{l_{x}}$$

to show how the last two columns of the Table were computed. In particular, by (4.31)

$$e_2 = \frac{100}{860} \cdot 0 + \frac{120}{860} \cdot 1 + \frac{140}{860} \cdot 2 + \frac{500}{860} \cdot 3 + \frac{1}{2} = \frac{1900}{860} + 0.5 = 2.709$$

Moreover: observe that  $c_x = \sum_{k=0}^{5-x} k_k p_x q_{x+k}$  satisfies the "recursion equation"  $c_x = p_x (1 + c_{x+1})$  (cf. Exercise 5 above), with  $c_5 = 0$ , from which the  $e_x$  column is easily computed by:  $e_x = c_x + 0.5$ .

Now apply the present value formula for conitunous insurance to find

$$\overline{A}_x = \sum_{k=0}^{5-x} {}_k p_x q_x v^k \frac{1-e^{-\delta}}{\delta} = 0.9582 \sum_{k=0}^{5-x} {}_k p_x q_x v^k = 0.9582 b_x$$

where  $b_x$  is the expected present value of an insurance of 1 payable at the beginning of the year of death (so that  $A_x = v b_x$ ) and satisfies  $b_5 = 1$  together with the recursion-relation

$$b_x = \sum_{k=0}^{5-x} {}_{k}p_x \, q_x \, v^k = p_x \, v \, b_{x+1} + q_x \tag{4.35}$$

(Proof of this recursion is Exercise 6 above.)

Example 2. Find a simplified expression in terms of actuarial exprected present value notations for the net single premium of an insurance on a life aged x, which pays  $F(k) = C \ \ddot{a}_{\overline{n-k}|}$  if death occurs at any exact ages between x+k and x+k+1, for  $k=0,1,\ldots,n-1$ , and interpret the result.

Let us begin with the interpretation: the beneficiary receives at the end of the year of death a lump-sum equal in present value to a payment stream of C annually beginning at the end of the year of death and terminating at the end of the C annually beginning at the end of the year of death and terminating at the end of the C annually beginning at the end of the year. This payment stream, if superposed upon

an n-year life annuity-immediate with annual payments C, would result in a certain payment of C at the end of policy years  $1, 2, \ldots, n$ . Thus the expected present value in this example is given by

$$C a_{\overline{n}} - C a_{\overline{x:n}} \tag{4.36}$$

Next we re-work this example purely in terms of analytical formulas. By formula (4.36), the net single premium in the example is equal to

$$\sum_{k=0}^{n-1} v^{k+1} {}_{k} p_{x} q_{x+k} C \ddot{\mathbf{a}}_{\overline{n-k+1}} = C \sum_{k=0}^{n-1} v^{k+1} {}_{k} p_{x} q_{x+k} \frac{1 - v^{n-k}}{d}$$

$$= \frac{C}{d} \left\{ \sum_{k=0}^{n-1} v^{k+1} {}_{k} p_{x} q_{x+k} - v^{n+1} \sum_{k=0}^{n-1} ({}_{k} p_{x} - {}_{k+1} p_{x}) \right\}$$

$$= \frac{C}{d} \left\{ A^{1}_{\overline{x:n}} - v^{n+1} (1 - {}_{n} p_{x}) \right\}$$

$$= \frac{C}{d} \left\{ A_{\overline{x:n}} - v^{n} {}_{n} p_{x} - v^{n+1} (1 - {}_{n} p_{x}) \right\}$$

and finally, by substituting expression (4.14) with m=1 for  $A_{\overline{x:n|}}$ , we have

$$\frac{C}{d} \left\{ 1 - d \ddot{\mathbf{a}}_{\overline{x:n}} - (1 - v) v^{n}_{n} p_{x} - v^{n+1} \right\}$$

$$= \frac{C}{d} \left\{ 1 - d \left( 1 + \mathbf{a}_{\overline{x:n}} - v^{n}_{n} p_{x} \right) - d v^{n}_{n} p_{x} - v^{n+1} \right\}$$

$$= \frac{C}{d} \left\{ v - d \mathbf{a}_{\overline{x:n}} - v^{n+1} \right\} = C \left\{ \frac{1 - v^{n}}{i} - \mathbf{a}_{\overline{x:n}} \right\}$$

$$= C \left\{ \mathbf{a}_{\overline{n}} - \mathbf{a}_{\overline{x:n}} \right\}$$

So the analytically derived answer agrees with the one intuitively arrived at in formula (4.36).

#### 4.5 Useful Formulas from Chapter 4

$$T_m = [Tm]/m$$

p. 99

$$P(T_m = x + \frac{k}{m} \mid T \ge x) = {}_{k/m}p_x - {}_{(k+1)/m}p_x = {}_{k/m}p_x \cdot {}_{1/m}q_{x+k/m}$$
p. 100

Term life annuity 
$$\mathbf{a}_{\overline{x:n}}^{(m)} = \ddot{\mathbf{a}}_{\overline{x:n+1/m}}^{(m)} - 1/m$$

p. 102

Endowment 
$$A_{\overline{x:n}|} = {}_{n}E_{x} = v^{n}{}_{n}p_{x}$$

p. 102

$$A_x^{(m)} - A^{(m)} \frac{1}{x:n} = v^n {}_n p_x \cdot A_{x+n}$$

p. 103

$$A_{\overline{x:n|}}^{(m)} = A^{(m)}_{\overline{x:n|}} + A^{(m)}_{\overline{x:n|}} = A^{(m)}_{\overline{x:n|}} +_n E_x$$

p. 104

$$\ddot{\mathbf{a}}_{\overline{x:n}}^{(m)} = \mathcal{E}_x \left( \frac{1 - v^{\min(T_m - x + 1/m, n)}}{d^{(m)}} \right) = \frac{1 - A_{\overline{x:n}}^{(m)}}{d^{(m)}}$$

p. 104

$$d^{(m)} \ddot{a}_{\overline{x:n}|}^{(m)} + A_{\overline{x:n}|}^{(m)} = 1$$

p. 104

$$A_{\overline{x:n}|}^{1} = \mathcal{E}_{x} \left( v^{T_{m}-x+1/m} \right) = \sum_{k=0}^{nm-1} v^{(k+1)/m} {}_{k/m} p_{x} {}_{1/m} q_{x+k/m}$$

p. 105

$$A_{\overline{x:n|}}^{1} = \sum_{k=0}^{n-1} v^{k+1} {}_{k} p_{x} q_{x+k}$$

p. 107

$$\ddot{\mathbf{a}}_{\overline{x:n}|} = \sum_{k=0}^{n-1} v^k_{k} p_x$$

p. 107

$$A_{\overline{x:n|}}^{1} = \mathcal{E}_{x} \Big( v^{n} I_{[T-x \geq n]} \Big) = v^{n} {}_{n} p_{x}$$

p. 107

$$A_{\overline{x:n}} = \sum_{k=0}^{n-1} v^{k+1} (_k p_x - {}_{k+1} p_x) + v^n {}_n p_x$$

p. 107

### Chapter 5

### Premium Calculation

This Chapter treats the most important topics related to the calculation of (risk) premiums for realistic insurance and annuity contracts. We begin by considering at length net single premium formulas for insurance and annuities, under each of three standard assumptions on interpolation of the survival function between integer ages, when there are multiple payments per year. One topic covered more rigorously here than elsewhere is the calculusbased and numerical comparison between premiums under these slightly different interpolation assumptions, justifying the standard use of the simplest of the interpolation assumptions, that deaths occur uniformly within whole years of attained age. Next we introduce the idea of calculating *level* premiums, setting up equations balancing the stream of level premium payments coming in to an insurer with the payout under an insurance, endowment, or annuity contract. Finally, we discuss single and level premium calculation for insurance contracts where the death benefit is modified by (fractional) premium amounts, either as refunds or as amounts still due. Here the issue is first of all to write an exact balance equation, then load it appropriately to take account of administrative expenses and the cushion required for insurance-company profitability, and only then to approximate and obtain the usual formulas.

#### 5.1 m-Payment Net Single Premiums

The objective in this section is to relate the formulas for net single premiums for life insurance, life annuities, pure endowments and endowment insurances in the case where there are multiple payment periods per year to the case where there is just one. Of course, we must now make some interpolation assumptions about within-year survival in order to do this, and we consider the three main assumptions previously introduced: piecewise uniform failure distribution (constant failure density within each year), piecewise exponential failure distribution (constant force of mortality within each year), and Balducci assumption. As a practical matter, it usually makes relatively little difference which of these is chosen, as we have seen in exercises and will illustrate further in analytical approximations and numerical tabulations. However, of the three assumptions, Balducci's is least important practically, because of the remark that the force of mortality it induces within years is actually decreasing (the reciprocal of a linear function with positive slope), since formula (3.9) gives it under that assumption as

$$\mu(x+t) = -\frac{d}{dt} \ln S(x+t) = \frac{q_x}{1 - (1-t) q_x}$$

Thus the inclusion of the Balducci assumption here is for completeness only, since it is a recurring topic for examination questions. However, we do not give separate net single premium formulas for the Balducci case.

In order to display simple formulas, and emphasize closed-form relationships between the net single premiums with and without multiple payments per year, we adopt a further restriction throughout this Section, namely that the duration n of the life insurance or annuity is an integer even though m > 1. There is in principle no reason why all of the formulas cannot be extended, one by one, to the case where n is assumed only to be an integer multiple of 1/m, but the formulas are less simple that way.

## 5.1.1 Dependence Between Integer & Fractional Ages at Death

One of the clearest ways to distinguish the three interpolation assumptions is through the probabilistic relationship they impose between the greatest-

integer [T] or attained integer age at death and the fractional age T - [T] at death. The first of these is a discrete, nonnegative-integer-valued random variable, and the second is a continuous random variable with a density on the time-interval [0,1). In general, the dependence between these random variables can be summarized through the calculated joint probability

$$P([T] = x + k, T - [T] < t \mid T \ge x) = \int_{x+k}^{x+k+t} \frac{f(y)}{S(x)} dy = {}_{t}q_{x+k} {}_{k}p_{x}$$
 (5.1)

where k, x are integers and  $0 \le t < 1$ . From this we deduce the following formula (for  $k \ge 0$ ) by dividing the formula (5.1) for general t by the corresponding formula at t = 1:

$$P(T - [T] \le t \mid [T] = x + k) = \frac{tq_{x+k}}{q_{x+k}}$$
(5.2)

where we have used the fact that T - [T] < 1 with certainty.

In case (i) from Section 3.2, with the density f assumed piecewise constant, we already know that  $_tq_{x+k}=t\,q_{x+k}$ , from which formula (5.2) immediately implies

$$P(T - [T] < t | [T] = x + k) = t$$

In other words, given complete information about the age at death, the fractional age at death is always uniformly distributed between 0, 1. Since the conditional probability does not involve the age at death, we say under the interpolation assumption (i) that the fractional age and whole-year age at death are *independent* as random variables.

In case (ii), with piecewise constant force of mortality, we know that

$$_{t}q_{x+k} = 1 - _{t}p_{x+k} = 1 - e^{-\mu(x+k)t}$$

and it is no longer true that fractional and attained ages at death are independent except in the very special (completely artificial) case where  $\mu(x+k)$  has the same constant value  $\mu$  for all x, k. In the latter case, where T is an *exponential* random variable, it is easy to check from (5.2) that

$$P(T - [T] \le t \mid [T] = x + k) = \frac{1 - e^{-\mu t}}{1 - e^{-\mu}}$$

In that case, T - [T] is indeed independent of [T] and has a truncated exponential distribution on [0,1), while [T] has the Geometric  $(1-e^{-\mu})$  distribution given, according to (5.1), by

$$P([T] = x + k \mid T \ge x) = (1 - e^{-\mu})(e^{-\mu})^k$$

In case (iii), under the Balducci assumption, formula (3.8) says that  $_{1-t}q_{x+t} = (1-t)q_x$ , which leads to a special formula for (5.2) but not a conclusion of conditional independence. The formula comes from the calculation

$$(1-t) q_{x+k} = {}_{(1-t)}q_{x+k+t} = 1 - \frac{p_{x+k}}{tp_{x+k}}$$

leading to

$$_{t}q_{x+k} = 1 - _{t}p_{x+k} = 1 - \frac{p_{x+k}}{1 - (1-t)q_{x+k}} = \frac{t q_{x+k}}{1 - (1-t)q_{x+k}}$$

Thus Balducci implies via (5.2) that

$$P(T - [T] \le t \mid [T] = x + k) = \frac{t}{1 - (1 - t) q_{x+k}}$$

#### 5.1.2 Net Single Premium Formulas — Case (i)

In this setting, the formula (4.15) for insurance net single premium is simpler than (4.16) for life annuities, because

$$_{j/m}p_{x+k} - _{(j+1)/m}p_{x+k} = \frac{1}{m}q_{x+k}$$

Here and throughout the rest of this and the following two subsections, x, k, j are integers and  $0 \le j < m$ , and  $k + \frac{j}{m}$  will index the possible values for the last multiple  $T_m - x$  of 1/m year of the policy age at death. The formula for net single insurance premium becomes especially simple when n is an integer, because the double sum over j and k factors into the product of a sum of terms depending only on j and one depending only on k:

$$A^{(m)1}_{\overline{x:n}} = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} v^{k+(j+1)/m} \frac{1}{m} q_{x+k} {}_{k} p_{x}$$

$$= \left(\sum_{k=0}^{n-1} v^{k+1} q_{x+k} k p_x\right) \frac{v^{-1+1/m}}{m} \sum_{j=0}^{m-1} v^{j/m} = A_{\overline{x:n}|}^1 v^{-1+1/m} \ddot{\mathbf{a}}_{\overline{1}|}^{(m)}$$
$$= A_{\overline{x:n}|}^1 v^{-1+1/m} \frac{1-v}{d^{(m)}} = \frac{i}{i^{(m)}} A_{\overline{x:n}|}^1$$
(5.3)

The corresponding formula for the case of non-integer n can clearly be written down in a similar way, but does not bear such a simple relation to the one-payment-per-year net single premium.

The formulas for life annuities should not be re-derived in this setting but rather obtained using the general identity connecting endowment insurances with life annuities. Recall that in the case of integer n the net single premium for a pure n-year endowment does not depend upon m and is given by

$$A_{\overline{x:n}}^{1} = {}_{n}p_{x} v^{n}$$

Thus we continue by displaying the net single premium for an endowment insurance, related in the *m*-payment-period-per year case to the formula with single end-of-year payments:

$$A_{\overline{x:n}|}^{(m)} = A^{(m)}_{\overline{x:n}|} + A_{\overline{x:n}|}^{1} = \frac{i}{i^{(m)}} A_{\overline{x:n}|}^{1} + {}_{n} p_{x} v^{n}$$
 (5.4)

As a result of (4.11), we obtain the formula for net single premium of a temporary life-annuity due:

$$\ddot{\mathbf{a}}_{\overline{x:n}|}^{(m)} = \frac{1 - A_{\overline{x:n}|}^{(m)}}{d^{(m)}} = \frac{1}{d^{(m)}} \left[ 1 - \frac{i}{i^{(m)}} A_{\overline{x:n}|}^{1} - {}_{n} p_{x} v^{n} \right]$$

Re-expressing this formula in terms of annuities on the right-hand side, using  $\ddot{a}_{x:\vec{n}|} = d^{-1} \left(1 - v^n p_x - A^1_{x:\vec{n}|}\right)$ , immediately yields

$$\ddot{\mathbf{a}}_{\overline{x:n}|}^{(m)} = \frac{d i}{d^{(m)} i^{(m)}} \ddot{\mathbf{a}}_{\overline{x:n}|} + \left(1 - \frac{i}{i^{(m)}}\right) \frac{1 - v^n {}_n p_x}{d^{(m)}}$$
(5.5)

The last formula has the form that the life-annuity due with m payments per year is a weighted linear combination of the life-annuity due with a single payment per year, the n-year pure endowment, and a constant, where the weights and constant depend only on interest rates and m but not on survival probabilities:

$$\ddot{\mathbf{a}}_{\overline{x:n}|}^{(m)} = \alpha(m) \ddot{\mathbf{a}}_{\overline{x:n}|} - \beta(m) (1 - {}_{n}p_{x}v^{n})$$

$$= \alpha(m) \ddot{\mathbf{a}}_{\overline{x:n}|} - \beta(m) + \beta(m) A_{\overline{x:n}|}^{-1}$$
(5.6)

i	m=	2	3	4	6	12
0.06						
0.03	3					
	$\alpha(m)$	1.0001	1.0001	1.0001	1.0001	1.0001
	eta(m)	0.2537	0.3377	0.3796	0.4215	0.4633
0.05	5					
	$\alpha(m)$	1.0002	1.0002	1.0002	1.0002	1.0002
	$\beta(m)$	0.2562	0.3406	0.3827	0.4247	0.4665
0.07	7					
	$\alpha(m)$	1.0003	1.0003	1.0004	1.0004	1.0004
	$\beta(m)$	0.2586	0.3435	0.3858	0.4278	0.4697
0.08	3					
	$\alpha(m)$	1.0004	1.0004	1.0005	1.0005	1.0005
	$\beta(m)$	0.2598	0.3450	0.3873	0.4294	0.4713
0.10	)					
	$\alpha(m)$	1.0006	1.0007	1.0007	1.0007	1.0008
	$\beta(m)$	0.2622	0.3478	0.3902	0.4325	0.4745
	,					

Table 5.1: Values of  $\alpha(m)$ ,  $\beta(m)$  for Selected m, i

Here the interest-rate related constants  $\alpha(m)$ ,  $\beta(m)$  are given by

$$\alpha(m) = \frac{d i}{d^{(m)} i^{(m)}}, \qquad \beta(m) = \frac{i - i^{(m)}}{d^{(m)} i^{(m)}}$$

Their values for some practically interesting values of m, i are given in Table 5.1. Note that  $\alpha(1)=1$ ,  $\beta(1)=0$ , reflecting that  $\ddot{\mathbf{a}}_{\overline{x:n}|}^{(m)}$  coincides with  $\ddot{\mathbf{a}}_{\overline{x:n}|}$  by definition when m=1. The limiting case for i=0 is given in Exercises 6 and 7:

for 
$$i = 0$$
,  $m \ge 1$ ,  $\alpha(m) = 1$ ,  $\beta(m) = \frac{m-1}{2m}$ 

Equations (5.3), (5.5), and (5.6) are useful because they summarize concisely the modification needed for one-payment-per-year formulas (which used only life-table and interest-rate-related quantities) to accommodate multiple payment-periods per year. Let us specialize them to cases where either

the duration n, the number of payment-periods m, or both approach  $\infty$ . Recall that failures continue to be assumed uniformly distributed within years of age.

Consider first the case where the insurances and life-annuities are wholelife, with  $n=\infty$ . The net single premium formulas for insurance and life annuity due reduce to

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x$$
 ,  $\ddot{a}_x^{(m)} = \alpha(m) \ddot{a}_x - \beta(m)$ 

Next consider the case where n is again allowed to be finite, but where m is taken to go to  $\infty$ , or in other words, the payments are taken to be instantaneous. Recall that both  $i^{(m)}$  and  $d^{(m)}$  tend in the limit to the force-of-interest  $\delta$ , so that the limits of the constants  $\alpha(m)$ ,  $\beta(m)$  are respectively

$$\alpha(\infty) = \frac{di}{\delta^2}$$
 ,  $\beta(\infty) = \frac{i - \delta}{\delta^2}$ 

Recall also that the instantaneous-payment notations replace the superscripts  $^{(m)}$  by an overbar. The single-premium formulas for instantaneous-payment insurance and life-annuities due become:

$$\overline{A}_{\overline{x:n}|}^{1} = \frac{i}{\delta} A_{\overline{x:n}|}^{1} , \qquad \overline{a}_{\overline{x:n}|} = \frac{d i}{\delta^{2}} \ddot{a}_{\overline{x:n}|} - \frac{i - \delta}{\delta^{2}} (1 - v^{n}_{n} p_{x})$$

## 5.1.3 Net Single Premium Formulas — Case (ii)

In this setting, where the force of mortality is constant within single years of age, the formula for life-annuity net single premium is simpler than the one for insurance, because for integers  $j, k \geq 0$ ,

$$_{k+j/m}p_x = _k p_x e^{-j\mu_{x+k}/m}$$

Again restrict attention to the case where n is a positive integer, and calculate from first principles (as in 4.16)

$$\ddot{\mathbf{a}}_{x:n}^{(m)} = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} \frac{1}{m} v^{k+j/m} {}_{j/m} p_{x+k} {}_{k} p_{x}$$

$$= \sum_{k=0}^{n-1} v^{k} {}_{k} p_{x} \sum_{j=0}^{m-1} \frac{1}{m} (v e^{-\mu_{x+k}})^{j/m} = \sum_{k=0}^{n-1} v^{k} {}_{k} p_{x} \frac{1 - v p_{x+k}}{m(1 - (v p_{x+k})^{1/m})}$$
(5.7)

where we have used the fact that when force of mortality is constant within years,  $p_{x+k} = e^{-\mu_{x+k}}$ . In order to compare this formula with equation (5.5) established under the assumption of uniform distribution of deaths within years of policy age, we apply the first-order Taylor series approximation about 0 for formula (5.7) with respect to the death-rates  $q_{x+k}$  inside the denominator-expression  $1 - (vp_{x+k})^{1/m} = 1 - (v - vq_{x+k})^{1/m}$ . (These annual death-rates  $q_{x+k}$  are actually small over a large range of ages for U.S. life tables.) The final expression in (5.7) will be Taylor-approximated in a slightly modified form: the numerator and denominator are both multiplied by the factor  $1 - v^{1/m}$ , and the term

$$(1-v^{1/m})/(1-(vp_{x+k})^{1/m})$$

will be analyzed first. The first-order Taylor-series approximation about z = 1 for the function  $(1 - v^{1/m})/(1 - (vz)^{1/m})$  is

$$\frac{1 - v^{1/m}}{1 - (vz)^{1/m}} \approx 1 - (1 - z) \left[ \frac{v^{1/m} (1 - v^{1/m}) z^{-1 + 1/m}}{m (1 - (vz)^{1/m})^2} \right]_{z=1}$$

$$= 1 - (1 - z) \frac{v^{1/m}}{m (1 - v^{1/m})} = 1 - \frac{1 - z}{i^{(m)}}$$

Evaluating this Taylor-series approximation at  $z = p_{x+k} = 1 - q_{x+k}$  then yields

$$\frac{1 - v^{1/m}}{1 - (vp_{x+k})^{1/m}} \approx 1 - \frac{q_{x+k}}{i^{(m)}}$$

Substituting this final approximate expression into equation (5.7), with numerator and denominator both multiplied by  $1 - v^{1/m}$ , we find for piecewise-constant force of mortality which is assumed small

$$\ddot{\mathbf{a}}_{\overline{x:n}|}^{(m)} \approx \sum_{k=0}^{n-1} v^k {}_k p_x \frac{1 - v p_{x+k}}{m(1 - v^{1/m})} \left(1 - q_{x+k}/i^{(m)}\right)$$

$$\approx \sum_{k=0}^{n-1} v^k {}_k p_x \frac{1}{d^{(m)}} \left\{1 - v p_{x+k} - \frac{1 - v}{i^{(m)}} q_{x+k}\right\}$$
(5.8)

where in the last line we have applied the identity  $m(1-v^{1/m}) = d^{(m)}$  and discarded a quadratic term in  $q_{x+k}$  within the large curly bracket.

We are now close to our final objective: proving that the formulas (5.5) and (5.6) of the previous subsection are in the present setting still valid as approximate formulas. Indeed, we now prove that the final expression (5.8) is precisely equal to the right-hand side of formula (5.6). The interest of this result is that (5.6) applied to piecewise-uniform mortality (Case (i)), while we are presently operating under the assumption of piecewise-constant hazards (Case ii). The proof of our assertion requires us to apply simple identities in several steps. First, observe that (5.8) is equal by definition to

$$\frac{1}{d^{(m)}} \left[ \ddot{\mathbf{a}}_{\overline{x:n}} - \mathbf{a}_{\overline{x:n}} - v^{-1} \frac{1-v}{i^{(m)}} A^{1}_{\overline{x:n}} \right]$$
 (5.9)

Second, apply the general formula for  $\ddot{a}_{x:n}$  as a sum to check the identity

$$\ddot{\mathbf{a}}_{\overline{x:n}} = \sum_{k=0}^{n-1} v^k{}_k p_x = 1 - v^n{}_n p_x + \mathbf{a}_{\overline{x:n}}$$
 (5.10)

and third, recall the identity

$$\ddot{\mathbf{a}}_{\overline{x:n}} = \frac{1}{d} \left( 1 - A_{\overline{x:n}}^{1} - v^{n}{}_{n} p_{x} \right)$$
 (5.11)

Substitute the identities (5.10) and (5.11) into expression (5.9) to re-express the latter as

$$\frac{1}{d^{(m)}} \left[ 1 - v^{n}_{n} p_{x} - \frac{i}{i^{(m)}} (1 - v^{n}_{n} p_{x} - d \ddot{a}_{\overline{x}:\overline{n}|}) \right] 
= \frac{d i}{d^{(m)} i^{(m)}} \ddot{a}_{\overline{x}:\overline{n}|} + \frac{1}{d^{(m)}} (1 - v^{n}_{n} p_{x}) (1 - \frac{i}{i^{(m)}})$$
(5.12)

The proof is completed by remarking that (5.12) coincides with expression (5.6) in the previous subsection.

Since formulas for the insurance and life annuity net single premiums can each be used to obtain the other when there are m payments per year, and since in the case of integer n, the pure endowment single premium  $A_{\overline{x:n}}^{-1}$  does not depend upon m, it follows from the result of this section that all of the formulas derived in the previous section for case (i) can be used as approximate formulas (to first order in the death-rates  $q_{x+k}$ ) also in case (ii).

# 5.2 Approximate Formulas via Case(i)

The previous Section developed a Taylor-series justification for using the very convenient net-single-premium formulas derived in case (i) (of uniform distribution of deaths within whole years of age) to approximate the corresponding formulas in case (ii) (constant force of mortality within whole years of age. The approximation was derived as a first-order Taylor series, up to linear terms in  $q_{x+k}$ . However, some care is needed in interpreting the result, because for this step of the approximation to be accurate, the year-by-year death-rates  $q_{x+k}$  must be small compared to the nominal rate of interest  $i^{(m)}$ . While this may be roughly valid at ages 15 to 50, at least in developed countries, this is definitely not the case, even roughly, at ages larger than around 55.

Accordingly, it is interesting to compare numerically, under several assumed death- and interest- rates, the individual terms  $A^{(m)} \frac{1}{x:k+1} - A^{(m)} \frac{1}{x:k}$  which arise as summands under the different interpolation assumptions. (Here and throughout this Section, k is an integer.) We first recall the formulas for cases (i) and (ii), and for completeness supply also the formula for case (iii) (the Balducci interpolation assumption). Recall that Balducci's assumption was previously faulted both for complexity of premium formulas and lack of realism, because of its consequence that the force of mortality decreases within whole years of age. The following three formulas are exactly valid under the interpolation assumptions of cases (i), (ii), and (iii) respectively.

$$A^{(m)} \frac{1}{x:k+1} - A^{(m)} \frac{1}{x:k} = \frac{i}{i^{(m)}} v^{k+1} {}_{k} p_{x} \cdot q_{x+k}$$
 (5.13)

$$A^{(m)} \frac{1}{x:k+1} - A^{(m)} \frac{1}{x:k} = v^{k+1} {}_{k} p_{x} \left(1 - p_{x+k}^{1/m}\right) \frac{i + q_{x+k}}{1 + (i^{(m)}/m) - p_{x+k}^{1/m}}$$
(5.14)

$$A^{(m)} \frac{1}{x:k+1} - A^{(m)} \frac{1}{x:k} = v^{k+1} {}_{k} p_{x} q_{x+k} \sum_{j=0}^{m-1} \frac{p_{x+k} v^{-j/m}}{m \left(1 - \frac{j+1}{m} q_{x+k}\right) \left(1 - \frac{j}{m} q_{x+k}\right)}$$

$$(5.15)$$

Formula (5.13) is an immediate consequence of the formula  $A^{(m)} = i A_{\overline{x:n}}^1 / i^{(m)}$  derived in the previous section. To prove (5.14), assume (ii) and calculate from first principles and the identities  $v^{-1/m} = 1 + i^{(m)}/m$  and

$$p_{x+k} = \exp(-\mu_{x+k}) \text{ that}$$

$$\sum_{j=0}^{m-1} v^{k+(j+1)/m} {}_k p_x \left( {}_{j/m} p_{x+k} - {}_{(j+1)/m} p_{x+k} \right)$$

$$= v^{k+1} {}_k p_x v^{-1+1/m} \left( 1 - e^{-\mu_{x+k}/m} \right) \sum_{j=0}^{m-1} \left( v e^{-\mu_{x+k}} \right)^{j/m}$$

$$= v^{k+1} {}_k p_x \left( 1 - e^{-\mu_{x+k}/m} \right) \frac{1 - v p_{x+k}}{1 - \left( v p_{x+k} \right)^{1/m}} \cdot \frac{v^{-1}}{v^{-1/m}}$$

$$= v^{k+1} {}_k p_x \left( 1 - e^{-\mu_{x+k}/m} \right) \frac{i + q_{x+k}}{1 + i^{(m)}/m} - p_{x+k}^{1/m}$$

Finally, for the Balducci case, (5.15) is established by calculating first

$$_{j/m}p_{x+k} = \frac{p_{x+k}}{1 - \frac{1-j/m}{q_{x+k+j/m}}} = \frac{p_{x+k}}{1 - \frac{m-j}{m}q_{x+k}}$$

Then the left-hand side of (5.15) is equal to

$$\sum_{j=0}^{m-1} v^{k+(j+1)/m} {}_{k} p_{x} \left( {}_{j/m} p_{x+k} - {}_{(j+1)/m} p_{x+k} \right)$$

$$= v^{k+1} {}_{k} p_{x} q_{x+k} v^{-1+1/m} \sum_{j=0}^{m-1} \frac{p_{x+k} v^{j/m}}{m \left( 1 - \frac{m-j}{m} q_{x+k} \right) \left( 1 - \frac{m-j-1}{m} q_{x+k} \right)}$$

which is seen to be equal to the right-hand side of (5.15) after the change of summation-index j' = m - j - 1.

Formulas (5.13), (5.14), and (5.15) are progressively more complicated, and it would be very desirable to stop with the first one if the choice of interpolation assumption actually made no difference. In preparing the following Table, the ratios both of formulas (5.14)/(5.13) and of (5.15)/(5.13) were calculated for a range of possible death-rates  $q = q_{x+k}$ , interest-rates i, and payment-periods-per-year m. We do not tabulate the results for the ratios (5.14)/(5.13) because these ratios were equal to 1 to three decimal places except in the following cases: the ratio was 1.001 when i ranged from 0.05 to 0.12 and q = 0.15 or when i was .12 or .15 and q was .12, achieving a value of 1.002 only in the cases where q = i = 0.15,  $m \ge 4$ .

Such remarkable correspondence between the net single premium formulas in cases (i), (ii) was by no means guaranteed by the previous Taylor series calculation, and is made only somewhat less surprising by the remark that the ratio of formulas (5.14)/(5.13) is smooth in both parameters  $q_{x+k}$ , i and exactly equal to 1 when either of these parameters is 0.

The Table shows a bit more variety in the ratios of (5.15)/(5.13), showing in part why the Balducci assumption is not much used in practice, but also showing that for a large range of ages and interest rates it also gives correct answers within 1 or 2 %. Here also there are many cases where the Balducci formula (5.15) agrees extremely closely with the usual actuarial (case (i)) formula (5.13). This also can be partially justified through the observation (a small exercise for the reader) that the ratio of the right-hand sides of formulas (5.15) divided by (5.13) are identical in either of the two limiting cases where i = 0 or where  $q_{x+k} = 0$ . The Table shows that the deviations from 1 of the ratio (5.15) divided by (5.13) are controlled by the parameter m and the interest rate, with the death-rate much less important within the broad range of values commonly encountered.

## 5.3 Net Level (Risk) Premiums

The general principle previously enunciated regarding equivalence of two different (certain) payment-streams if their present values are equal, has the following extension to the case of uncertain (time-of-death-dependent) payment streams: two such payment streams are equivalent (in the sense of having equal 'risk premiums') if their expected present values are equal. This definition makes sense if each such equivalence is regarded as the matching of random income and payout for the insurer with respect to each of a large number of independent (and identical) policies. Then the **Law of Large Numbers** has the interpretation that the actual random net payout minus income for the aggregate of the policies per policy is with very high probability very close (percentagewise) to the mathematical expectation of the difference between the single-policy payout and income. That is why, from a pure-risk perspective, before allowing for administrative expenses and the 'loading' or cushion which an insurer needs to maintain a very tiny probability of going bankrupt after starting with a large but fixed fund of reserve

Table 5.2: Ratios of Values (5.15)/(5.13)

$q_{x+k}$	i	m=2	m=4	m=12
.002	.03	1.015	1.007	1.002
.006	.03	1.015	1.007	1.002
.02	.03	1.015	1.008	1.003
.06	.03	1.015	1.008	1.003
.15	.03	1.015	1.008	1.003
.002	.05	1.025	1.012	1.004
.006	.05	1.025	1.012	1.004
.02	.05	1.025	1.012	1.004
.06	.05	1.025	1.013	1.005
.15	.05	1.026	1.014	1.005
.002	.07	1.034	1.017	1.006
.006	.07	1.034	1.017	1.006
.02	.07	1.035	1.017	1.006
.06	.07	1.035	1.018	1.006
.15	.07	1.036	1.019	1.007
.002	.10	1.049	1.024	1.008
.006	.10	1.049	1.024	1.008
.02	.10	1.049	1.024	1.008
.06	.10	1.050	1.025	1.009
.15	.10	1.051	1.027	1.011
.002	.12	1.058	1.029	1.010
.006	.12	1.058	1.029	1.010
.02	.12	1.059	1.029	1.010
.06	.12	1.059	1.030	1.011
.15	.12	1.061	1.032	1.013
.002	.15	1.072	1.036	1.012
.006	.15	1.072	1.036	1.012
.02	.15	1.073	1.036	1.012
.06	.15	1.074	1.037	1.013
.15	.15	1.075	1.039	1.016

capital, this expected difference should be set equal to 0 in figuring premiums. The resulting rule for calculation of the premium amount P which must multiply the unit amount in a specified payment pattern is as follows:

 $P = \text{Expected present value of life insurance, annuity, or endowment contract proceeds divided by the expected present value of a unit amount paid regularly, according to the specified payment pattern, until death or expiration of term.$ 

## 5.4 Benefits Involving Fractional Premiums

The general principle for calculating risk premiums sets up a balance between expected payout by an insurer and expected payment stream received as premiums. In the simplest case of level payment streams, the insurer receives a life-annuity due with level premium P, and pays out according to the terms of the insurance product purchased, say a term insurance. If the insurance purchased pays only at the end of the year of death, but the premium payments are made m times per year, then the balance equation becomes

$$A^{1}_{\overline{x:n}} = P \cdot m \ddot{\mathbf{a}}_{\overline{x:n}}^{(m)}$$

for which the solution P is called the *level risk premium for a term insurance*. The reader should distinguish this premium from the level premium payable m times yearly for an insurance which pays at the end of the  $(1/m)^{th}$  year of death. In the latter case, where the number of payment periods per year for the premium agrees with that for the insurance, the balance equation is

$$A^{(m)1}_{\overline{x:n}} = P \cdot m \ddot{\mathbf{a}}_{\overline{x:n}}^{(m)}$$

In standard actuarial notations for premiums, not given here, level premiums are annualized (which would result in the removal of a factor m from the right-hand sides of the last two equations).

Two other applications of the balancing-equation principle can be made in calculating level premiums for insurances which either (a) deduct the additional premium payments for the remainder of the year of death from the insurance proceeds, or (b) refund a pro-rata share of the premium for the portion of the 1/m year of death from the instant of death to the end of the 1/m year of death. Insurance contracts with provision (a) are called insurances with *installment* premiums: the meaning of this term is that the insurer views the full year's premium as due at the beginning of the year, but that for the convenience of the insured, payments are allowed to be made in installments at m regularly spaced times in the year. Insurances with provision (b) are said to have apportionable refund of premium, with the implication that premiums are understood to cover only the period of the year during which the insured is alive. First in case (a), the expected amount paid out by the insurer, if each level premium payment is P and the face amount of the policy is F(0), is equal to

$$F(0) A^{(m)} \frac{1}{x:n} - \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} v^{k+(j+1)/m} \frac{1}{k+j/m} p_x \cdot \frac{1}{m} q_{x+k+j/m} (m-1-j) P$$

and the exact balance equation is obtained by setting this equal to the expected amount paid in, which is again  $Pm\ddot{a}_{\overline{x:n}}^{(m)}$ . Under the interpolation assumption of case (i), using the same reasoning which previously led to the simplified formulas in that case, this balance equation becomes

$$F(0) A^{(m)} \frac{1}{x:n} - A^{1}_{\overline{x:n}} \frac{P}{m} \sum_{j=0}^{m-1} v^{-(m-j-1)/m} (m-j-1) = P m \ddot{a}_{\overline{x:n}}^{(m)}$$
 (5.16)

Although one could base an exact calculation of P on this equation, a further standard approximation leads to a simpler formula. If the term (m-j-1) is replaced in the final sum by its average over j, or by  $m^{-1} \sum_{j=0}^{m-1} (m-j-1) = m^{-1} (m-1)m/2 = (m-1)/2$ , we obtain the installment premium formula

$$P = \frac{F(0) A^{(m)} \frac{1}{x:n|}}{m \ddot{a}_{x:n|}^{(m)} + \frac{m-1}{2} A^{(m)} \frac{1}{x:n|}}$$

and this formula could be related using previous formulas derived in Section 5.1 to the insurance and annuity net single premiums with only one payment period per year.

In the case of the apportionable return of premium, the only assumption usually considered is that of case (i), that the fraction of a single premium payment which will be returned is on average 1/2 regardless of which of

the 1/m fractions of the year contains the instant of death. The balance equation is then very simple:

$$A^{(m)1}_{\overline{x:n}} \left( F(0) + \frac{1}{2} P \right) = P m \ddot{\mathbf{a}}_{\overline{x:n}}^{(m)}$$
 (5.17)

and this equation has the straightforward solution

$$P = \frac{F(0) A^{(m)}_{\overline{x:n}|}}{m \ddot{a}_{\overline{x:n}|}^{(m)} - \frac{1}{2} A^{(m)}_{\overline{x:n}|}}$$

It remains only to remark what is the effect of loading for administrative expenses and profit on insurance premium calculation. If all amounts paid out by the insurer were equally loaded (i.e., multiplied) by the factor 1 + L, then formula (5.17) would involve the loading in the second term of the denominator, but this is apparently not the usual practice. In both the apportionable refund and installment premium contracts, as well as the insurance contracts which do not modify proceeds by premium fractions, it is apparently the practice to load the level premiums P directly by the factor 1+L, which can easily be seen to be equivalent to inflating the face-amount F(0) in the balance-formulas by this factor.

#### **5.5** Exercise Set 5

(1). Show from first principles that for all integers x, n, and all fixed interest-rates and life-distributions

$$a_{\overline{x:n|}} = \ddot{\mathbf{a}}_{\overline{x:n|}} - 1 + v^n {}_n p_x$$

(2). Show from first principles that for all integers x, and all fixed interestrates and life-distributions

$$A_x = v \ddot{\mathbf{a}}_x - a_x$$

Show further that this relation is obtained by taking the expectation on both sides of an identity in terms of present values of payment-streams, an identity whoch holds for each value of (the greatest integer [T] less than or equal to) the exact-age-at-death random variable T.

(3). Using the same idea as in problem (2), show that (for all x, n, interest rates, and life-distributions)

$$A_{\overline{x:n}}^{1} = v \ddot{\mathbf{a}}_{\overline{x:n}} - a_{\overline{x:n}}$$

- (4). Suppose that a life aged x (precisely, where x is an integer) has the survival probabilities  $p_{x+k} = 0.98$  for  $k = 0, 1, \ldots, 9$ . Suppose that he wants to purchase a term insurance which will pay \$30,000 at the end of the quarter-year of death if he dies within the first five years, and will pay \$10,000 (also at the end of the quarter-year of death) if he dies between exact ages 5, 10. In both parts (a), (b) of the problem, assume that the interest rate is fixed at 5%, and assume wherever necessary that the individual's distribution of death-time is uniform within each whole year of age.
  - (a) Find the net single premium of the insurance contract described.
- (b) Suppose that the individual purchasing the insurance described wants to pay level premiums semi-annually, beginning immediately. Find the amount of each semi-annual payment.
- (5). Re-do problem (4) assuming in place of the uniform distribution of age at death that the insured individual has constant force of mortality within each whole year of age. Give your numerical answers to at least 6 significant figures so that you can compare the exact numerical answers in these two problems.
- (6). Using the exact expression for the interest-rate functions  $i^{(m)}$ ,  $d^{(m)}$  respectively as functions of i and d, expand these functions in Taylor series about 0 up to quadratic terms. Use the resulting expressions to approximate the coefficients  $\alpha(m)$ ,  $\beta(m)$  which were derived in the Chapter. Hence justify the so-called *traditional approximation*

$$\ddot{\mathbf{a}}_x^{(m)} \approx \ddot{\mathbf{a}}_x - \frac{m-1}{2m}$$

(7). Justify the 'traditional approximation' (the displayed formula in Exercise 6) as an exact formula in the case (i) in the limit  $i \to 0$ , by filling in the details of the following argument.

No matter which policy-year is the year of death of the annuitant, the policy with m = 1 (and expected present value  $\ddot{a}_x$ ) pays 1 at the beginning

of that year while the policy with m > 1 pays amounts 1/m at the beginning of each 1/m'th year in which the annuitant is alive. Thus, the annuity with one payment per years pays more than the annuity with m > 1 by an absolute amount  $1 - (T_m - [T] + 1/m)$ . Under assumption (i),  $T_m - [T]$  is a discrete random variable taking on the possible values  $0, 1, \ldots, (m-1)/m$  each with probability 1/m. Disregard the interest and present-value discounting on the excess amount  $1 - (T_m - [T])/m$  paid by the m-payment-per year annuity, and show that it is exactly (m-1)/2m.

- (8). Give an exact formula for the error of the 'traditional approximation' given in the previous problem, in terms of m, the constant interest rate i (or  $v = (1+i)^{-1}$ ), and the constant force  $\mu$  of mortality, when the lifetime T is assumed to be distributed precisely as an  $Exponential(\mu)$  random variable.
- (9). Show that the ratio of formulas (5.14)/(5.13) is 1 whenever either  $q_{x+k}$  or i is set equal to 0.
- (10). Show that the ratio of formulas (5.15)/(5.13) is 1 whenever either  $q_{x+k}$  or i is set equal to 0.
- (11). For a temporary life annuity on a life aged 57, with benefits deferred for three years, you are given that  $\mu_x = 0.04$  is constant,  $\delta = .06$ , that premiums are paid continuously (with  $m = \infty$ ) only for the first two years, at rate  $\overline{P}$  per year, and that the annuity benefits are payable at beginnings of years according to the following schedule:

- (a) In terms of  $\overline{P}$ , calculate the expected present value of the premiums paid.
- (b) Using the equivalence principle, calculate  $\overline{P}$  numerically.
- (12). You are given that (i)  $q_{60} = 0.3$ ,  $q_{61} = 0.4$ , (ii) f denotes the probability that a life aged 60 will die between ages 60.5 and 61.5 under the assumption of uniform distribution of deaths within whole years of age, and (iii) g denotes the probability that a life aged 60 will die between ages 60.5 and 61.5 under the Balducci assumption. Calculate  $10,000 \cdot (g-f)$  numerically, with accuracy to the nearest whole number.

(13). You are given that S(40) = 0.500, S(41) = 0.475, i = 0.06,  $\overline{A}_{41} = 0.54$ , and that deaths are uniformly distributed over each year of age. Find  $A_{40}$  exactly.

(14). If a mortality table follows Gompertz' law (with exponent c), prove that

$$\mu_x = \overline{A}_x / \overline{a}_x'$$

where  $\overline{A}_x$  is calculated at interest rate i while  $\overline{a}'_x$  is calculated at a rate of interest  $i' = \frac{1+i}{c} - 1$ .

(15). You are given that i = 0.10,  $q_x = 0.05$ , and  $q_{x+1} = 0.08$ , and that deaths are uniformly distributed over each year of age. Calculate  $\overline{A}_{\overline{x}:2}^1$ .

(16). A special life insurance policy to a life aged x provides that if death occurs at any time within 15 years, then the only benefit is the return of premiums with interest compounded to the end of the year of death. If death occurs after 15 years, the benefit is \$10,000. In either case, the benefit is paid at the end of the year of death. If the premiums for this policy are to be paid yearly for only the first 5 years (starting at the time of issuance of the policy), then find a simplified expression for the level annual pure-risk premium for the policy, in terms of standard actuarial and interest functions.

(17). Prove that for every m, n, x, k, the net single premium for an n-year term insurance for a life aged x, with benefit deferred for k years, and payable at the end of the 1/m year of death is given by either side of the identity

$$A_{\overline{x}|}^{n+k m} - A_{\overline{x}|}^{k m} = {}_{k}E_{x} A_{\overline{x+k}|}^{n m}$$

First prove the identity algebraically; then give an alternative, intuitive explanation of why the right-hand side represents the expected present value of the same contingent payment stream as the left-hand side.

## 5.6 Worked Examples

Overview of Premium Calculation for Single-Life Insurance & Annuities

Here is a schematic overview of the calculation of net single and level premiums for life insurances and life annuities, based on life-table or theoretical survival probabilities and constant interest or discount rate. We describe the general situation and follow a specific case study/example throughout.

(I) First you will be given information about the constant assumed interest rate in any of the equivalent forms  $i^{(m)}$ ,  $d^{(m)}$ , or  $\delta$ , and you should immediately convert to find the effective annual interest rate (APR) i and one-year discount factor v = 1/(1+i). In our case-study, assume that

the force of interest  $\delta$  is constant  $= -\ln(0.94)$ 

so that  $i = \exp(\delta) - 1 = (1/0.94) - 1 = 6/94$ , and v = 0.94. In terms of this quantity, one immediately answers a question such as "what is the present value of \$1 at the end of  $7\frac{1}{2}$  years?" by:  $v^{7.5}$ .

(II) Next you must be given **either** a theoretical survival function for the random age at death of a life aged x, in any of the equivalent forms S(x+t),  $_tp_x$ , f(x+t), or  $\mu(x+t)$ , **or** a cohort-form life-table, e.g.,

$$\begin{array}{llll} l_{25} = & 10,000 & & _{0}p_{25} = 1.0 \\ l_{26} = & 9,726 & & _{1}p_{25} = 0.9726 \\ l_{27} = & 9,443 & & _{2}p_{25} = 0.9443 \\ l_{28} = & 9,137 & & _{3}p_{25} = 0.9137 \\ l_{29} = & 8,818 & & _{4}p_{25} = 0.8818 \\ l_{30} = & 8,504 & & _{5}p_{25} = 0.8504 \end{array}$$

From such data, one calculates immediately that (for example) the probability of dying at an odd attained-age between 25 and 30 inclusive is

$$(1 - 0.9726) + (0.9443 - 0.9137) + (0.8818 - 0.8504) = 0.0894$$

The generally useful additional column to compute is:

$$q_{25} = 1 - {}_{1}p_{25} = 0.0274, \quad {}_{1}p_{25} - {}_{2}p_{25} = 0.0283, \quad {}_{2}p_{25} - {}_{3}p_{25} = 0.0306$$

$$_{3}p_{25} - _{4}p_{25} = 0.0319, \quad _{4}p_{25} - _{5}p_{25} = 0.0314$$

(III) In any problem, the terms of the life insurance or annuity to be purchased will be specified, and you should re-express its present value in terms of standard functions such as  $\mathbf{a}_{\overline{x:n}}$  or  $A^1_{\overline{x:n}}$ . For example, suppose a life aged x purchases an endowment/annuity according to which he receives \$10,000 once a year starting at age x+1 until either death occurs or n years have elapsed, and if he is alive at the end of n years he receives \$15,000. This contract is evidently a superposition of a n-year pure endowment with face value \$15,000 and a n-year temporary life annuity-immediate with yearly payments \$10,000. Thus, the expected present value (= net single premium) is

$$10,000 \, a_{\overline{x:n}} + 15,000 \, {}_{n}p_{x} \, v^{n}$$

In our case-study example, this expected present value is

$$= 10000 \left( 0.94(0.9726) + 0.94^{2}(0.9443) + 0.94^{3}(0.9137) + 0.94^{4}(0.8818) + 0.94^{5}(0.8504) \right) + 15000(0.94^{5} \cdot 0.8504)$$

The annuity part of this net single premium is \$38,201.09, and the pure-endowment part is \$9,361.68, for a total net single premium of \$47,562.77

(IV) The final part of the premium computation problem is to specify the type of payment stream with which the insured life intends to pay for the contract whose expected present value has been figured in step (III). If the payment is to be made at time 0 in one lump sum, then the net single premium has already been figured and we are done. If the payments are to be constant in amount (level premiums), once a year, to start immediately, and to terminate at death or a maximum of n payments, then we divide the net single premium by the expected present value of a unit life annuity  $\ddot{a}_{\overline{x:n}|}$ . In general, to find the premium we divide the net single premium of (III) by the expected present value of a unit amount paid according to the desired premium-payment stream.

In the case-study example, consider two cases. The first is that the purchaser aged x wishes to pay in two equal installments, one at time 0 and one after 3 years (with the second payment to be made only if he is alive at that time). The expected present value of a unit amount paid in this fashion is

$$1 + v^3_{3}p_x = 1 + (0.94)^3_{3} = 0.9137 = 1.7589$$

Thus the premium amount to be paid at each payment time is

$$\$47,563 / 1.7589 = \$27,041$$

Alternatively, as a second example, suppose that the purchaser is in effect taking out his annuity/endowment in the form of a loan, and agrees to (have his estate) repay the loan unconditionally (i.e. without regard to the event of his death) over a period of 29 years, with 25 equal payments to be made every year beginning at the end of 5 years. In this case, no probabilities are involved in valuing the payment stream, and the present value of such a payment stream of unit amounts is

$$v^4 a_{\overline{25}} = (0.94)^4 (.94/.06) (1 - (0.94)^{25}) = 9.627$$

In this setting, the amount of each of the equal payments must be

$$\$47,563/9.627 = \$4941$$

(V) To complete the circle of ideas given here, let us re-do the casestudy calculation of paragraphs (III) to cover the case where the insurance has quarterly instead of annual payments. Throughout, assume that deaths within years of attained age are uniformly distributed (case(i)).

First, the expected present value to find becomes

$$10,000 \, \mathbf{a}_{\overline{x:n}|}^{(4)} + 15,000 \, A_{\overline{x:n}|}^{1} = 10000 \left( \ddot{\mathbf{a}}_{\overline{x:n}|}^{(4)} - \frac{1}{4} \left( 1 - v^{n} _{n} p_{x} \right) \right) + 15000 \, v^{n} _{n} p_{x}$$

which by virtue of (5.6) is equal to

$$= 10000 \alpha(4) \ddot{\mathbf{a}}_{\overline{x:n}} - (1 - v^n_n p_x) (10000 \beta(4) + 2500) + 15000 v^n_n p_x$$

In the particular case with v = 0.94, x = 25, n = 5, and cohort life-table given in (II), the net single premium for the endowment part of the contract has exactly the same value \$9361.68 as before, while the annuity part now has the value

$$10000 (1.0002991) (1 + 0.94(0.9726) + 0.94^{2}(0.9443) + 0.94^{3}(0.9137) + + 0.94^{4}(0.8818)) - (6348.19) (1 - 0.94^{5}(0.8504)) = 39586.31$$

Thus the combined present value is 48947.99: the increase of 1385 in value arises mostly from the earlier annuity payments: consider that the interest on the annuity value for one-half year is  $38201(0.94^{-0.5}-1)=1200$ .

## 5.7 Useful Formulas from Chapter 5

$$P([T] = x + k, T - [T] < t \mid T \ge x) = \int_{x+k}^{x+k+t} \frac{f(y)}{S(x)} dy = {}_{t}q_{x+k} {}_{k}p_{x}$$
p. 127

$$P(T - [T] \le t \mid [T] = x + k) = \frac{tq_{x+k}}{q_{x+k}}$$
 p. 127

$$A^{(m)1}_{\overline{x:n|}} = \frac{i}{i^{(m)}} A^1_{\overline{x:n|}}$$
 under (i) p. 129

$$\ddot{\mathbf{a}}_{\overline{x:n}|}^{(m)} = \frac{1 - A_{\overline{x:n}|}^{(m)}}{d^{(m)}} = \frac{1}{d^{(m)}} \left[ 1 - \frac{i}{i^{(m)}} A_{\overline{x:n}|}^{1} - {}_{n} p_{x} v^{n} \right]$$
p. 129

$$\ddot{\mathbf{a}}_{x:n|}^{(m)} = \frac{di}{d^{(m)}i^{(m)}} \ddot{\mathbf{a}}_{x:n|} + \left(1 - \frac{i}{i^{(m)}}\right) \frac{1 - v^n {}_n p_x}{d^{(m)}} \quad \text{under (i)}$$
p. 129

$$\ddot{\mathbf{a}}_{\overline{x:n|}}^{(m)} = \alpha(m) \ \ddot{\mathbf{a}}_{\overline{x:n|}} - \beta(m) \left(1 - {}_n p_x \, v^n\right) \quad \text{under (i)}$$
p. 129

$$\alpha(m) \; = \; \frac{d\,i}{d^{(m)}\,i^{(m)}} \;\;, \qquad \beta(m) \; = \; \frac{i\,-\,i^{(m)}}{d^{(m)}\,i^{(m)}}$$
p. 130

$$\alpha(m) = 1$$
 ,  $\beta(m) = \frac{m-1}{2m}$  when  $i = 0$  p. 130

$$\ddot{a}_{x:n}^{(m)} = \sum_{k=0}^{n-1} v^k {}_k p_x \frac{1 - v p_{x+k}}{m(1 - (v p_{x+k})^{1/m})} \quad \text{under (ii)}$$

p. 131

$$A^{(m)}_{\frac{1}{x:k+1}} - A^{(m)}_{\frac{1}{x:k}} = \frac{i}{i^{(m)}} v^{k+1} {}_k p_x \cdot q_{x+k} \quad \text{under (i)}$$
 p. 134

$$A^{(m)} \frac{1}{x:k+1} - A^{(m)} \frac{1}{x:k} = v^{k+1} {}_{k} p_{x} \left(1 - p_{x+k}^{1/m}\right) \frac{i + q_{x+k}}{1 + \left(i^{(m)}/m\right) - p_{x+k}^{1/m}} \quad \text{under (ii)}$$

p. 134

$$A^{(m)} \frac{1}{x:k+1} - A^{(m)} \frac{1}{x:k} = v^{k+1} {}_{k} p_{x} q_{x+k} \sum_{j=0}^{m-1} \frac{p_{x+k} v^{-j/m}}{m \left(1 - \frac{j+1}{m} q_{x+k}\right) \left(1 - \frac{j}{m} q_{x+k}\right)}$$
under (iii)

p. 134

Level Installment Risk Premium 
$$= \frac{F(0) \, A^{(m)} \frac{1}{x:n|}}{m \ddot{\mathbf{a}}_{\overline{x:n}|}^{(m)} \, + \, \frac{m-1}{2} \, A^{(m)} \frac{1}{x:n|}}$$

p. 139

Apportionable Refund Risk Premium 
$$= \frac{F(0) A^{(m)} \frac{1}{x:n|}}{m \ddot{a}_{x:n|}^{(m)} - \frac{1}{2} A^{(m)} \frac{1}{x:n|}}$$
p. 140

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