Survey of Affine Deligne-Lusztig Varieties

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Outline

1. A Question in $\sigma$-linear algebra
2. Basic Questions about ADLVs
3. Isocrystals and Mazur’s inequality
4. Non-emptiness of ADLVs in the affine Grassmannian
5. Dimensions of ADLVs in the affine Grassmannian
6. ADLVs in the affine flag variety
A question in $\sigma$-linear algebra

- Let $k = \mathbb{F}_q$. $\text{Gal}(\overline{k}/k)$ has a canonical generator $\sigma : x \mapsto x^q$.
- Let $\mathcal{O} := \overline{k}[[\epsilon]]$ and $\text{Frac}(\mathcal{O}) = L = \overline{k}((\epsilon))$. The Frobenius automorphism $\sigma$ of $L$ is defined by
  $$\sigma\left(\sum_i a_i \epsilon^i\right) = \sum_i a_i^q \epsilon^i.$$
- We have $L^\sigma = F := k((\epsilon))$ and $\mathcal{O}^\sigma = \mathcal{O}_F := k[[\epsilon]]$.
- $\sigma$-Linear Algebra Question: Given $b \in \text{GL}_n(L)$ and $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$, does there exist an $\mathcal{O}$-lattice $\Lambda \subset L^n$ such that $b \sigma(\Lambda) \subseteq \Lambda$, and
  $$\Lambda / b \sigma(\Lambda) \cong \mathcal{O} / \epsilon^{\mu_1} \oplus \cdots \oplus \mathcal{O} / \epsilon^{\mu_n},$$
in other words, such that $\text{inv}(\Lambda, b \sigma(\Lambda)) = \mu$? If yes, what is the dimension of the “space of such $\Lambda$’s”?
- Goal: Explain why this question is interesting and how it is answered.
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Examples

- Define $X_{\mu}^{\text{GL}_n}(b) = \{ \Lambda \subset L^n \mid \text{inv}(\Lambda, b\sigma(\Lambda)) = \mu \}$. Call it the **Affine Deligne-Lusztig Variety (ADLV)** associated to $\text{GL}_n$, $b$, and $\mu$.

- (I) $n = 2$, $b = 1$, and $\mu = (0, 0)$. Then

  
  $X_{\mu}^{\text{GL}_2}(b) = \{ \Lambda \mid \sigma(\Lambda) = \Lambda \}$

  $= \{ \mathcal{O}_F\text{-lattices } \Lambda_F \subset F^2 \}$

  $= \text{the vertices in the building (a tree) for } \text{GL}_2(F)$. This is an infinite discrete set ($\dim = 0$).

- (II) $n = 2$, $b = 1$ and $\mu = (\mu_1, \mu_2)$, where $\mu_1 \geq \mu_2$. Then

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Let $\Lambda_0 = \mathcal{O} e_1 \oplus \mathcal{O} e_2$. Let $K = \text{GL}_2(\mathcal{O}) = \text{Stab}_{\text{GL}_2(L)}(\Lambda_0)$.

Write $\Lambda = g \Lambda_0$ for $g \in \text{GL}_2(L)$.

Theory of elementary divisors implies

$$\Lambda \in X_{\mu}^{\text{GL}_2}(1) \iff g^{-1} \sigma(g) \in K \begin{bmatrix} \epsilon^{\mu_1} & 0 \\ 0 & \epsilon^{\mu_2} \end{bmatrix} K.$$ 

Taking determinants, the above implies

$$\epsilon^{\mu_1+\mu_2} \in \det(g^{-1} \sigma(g)) \mathcal{O}^x = \mathcal{O}^x,$$

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ADLVs for general $G$

- Let $G$ denote a (split) connected reductive group, and put $K = G(O)$.
- Examples: $GL_n$, $SL_n$, $SO(n)$, $Sp(2n)$, $G_2$, $E_8$, etc.
- The analog of $\mu = (\mu_1, \ldots, \mu_2) \in \mathbb{Z}^n$, with $\mu_1 \geq \cdots \geq \mu_n$ is a dominant cocharacter $\mu : \mathbb{G}_m \to A$, for $A$ a (split) maximal torus in $G$. Denote these by $X_\ast(A)_{\text{dom}}$.
- Cartan Decomposition: $G(L) = \bigsqcup_{\mu \in X_\ast(A)_{\text{dom}}} K \mu(\epsilon) K$.
- Define $X_\mu^G(b) = \{ gK \in G(L)/K \mid g^{-1}b\sigma(g) \in K\mu(\epsilon)K \}$.
- This is a locally closed, finite-dimensional subvariety of the affine Grassmannian $G'(L)/K$. 
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Classical Deligne-Lusztig varieties

- Let $B \subset G$ be a Borel subgroup containing $A$, and let $W = N_G(A)/A$ be the Weyl group.
- **Bruhat Decomposition** $G = \coprod_{w \in W} BwB$, where $G = G(\overline{k})$ and $B = B(\overline{k})$ here.
- Define $X_w = \{gB \in G/B \mid g^{-1}\sigma(g) \in BwB\}$.
- This is a locally closed subvariety of the flag variety $G/B$ which is non-empty, smooth, and has dimension equal to $\ell(w)$.
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Classical Deligne-Lusztig varieties

- Let $B \subset G$ be a Borel subgroup containing $A$, and let $W = N_G(A)/A$ be the Weyl group.

- **Bruhat Decomposition** $G = \bigsqcup_{w \in W} BwB$, where $G = G(\overline{k})$ and $B = B(\overline{k})$ here.

- Define $X_w = \{ gB \in G/B \mid g^{-1}\sigma(g) \in BwB \}$.

- This is a locally closed subvariety of the flag variety $G/B$ which is **non-empty**, **smooth**, and has **dimension** equal to $\ell(w)$.

- Deligne and Lusztig introduced these and they are a crucial tool in the representation theory of the finite groups of Lie type, i.e., the finite groups $G(\mathbb{F}_q)$. 
Basic Questions about ADLVs

- (I) For which \((\mu, b)\) is \(X^G_\mu(b) \neq \emptyset\)?

- (II) If non-empty, is \(X^G_\mu(b)\) equidimensional, and is there a formula for its dimension?

- (III) What is the geometric structure of \(X^G_\mu(b)\) (irreducible components, singularities, etc.)?

The fact that \(X^G_\mu(b)\) can be empty should be contrasted with the classical case.

Also, there are many different "Frobenius elements" \(b_\sigma\) (in the classical case there is only one, so only \(b = 1\) appears).

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Isocrystals

- **Usual context is** $p$-adic: $F = \mathbb{Q}_p$, $\mathcal{O}_F = \mathbb{Z}_p$, $L = \widehat{\mathbb{Q}}_{p}^{\text{un}}$, $\mathcal{O} = \text{ring of integers in } L$, $k = \mathbb{F}_p = \mathcal{O}_F/p\mathcal{O}_F$.

- $\sigma$ is the Frobenius automorphism: either $x \mapsto x^p \in \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$, or as the element of $\text{Gal}(L/F)$, defined by

$$
\sigma\left(\sum_{i \gg -\infty} a_ip^i\right) = \sum_{i \gg -\infty} a_ip^{p^i}.
$$

- An **isocrystal** is a pair $(V, \Phi)$, where $V$ is a finite-dimensional $L$-vector space, and $\Phi : V \rightarrow V$ is a $\sigma$-linear bijection:

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\Phi(\alpha v) = \sigma(\alpha)\Phi(v), \quad \forall v \in V, \quad \alpha \in L.
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- If $V_0$ is an $F$-vector space and $V = V_0 \otimes_F L$, then all $(V, \Phi)$ are of form $(V, b(1 \otimes \sigma))$, for $b \in \text{GL}(V) = \text{GL}_n(L)$.
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Dieudonne’s classification of isocrystals

Dieudonne proved that the category of isocrystals is abelian and semi-simple. The simple objects, parametrized by $\lambda = r/s \in \mathbb{Q}$, are of form

$$V_\lambda := (L^s, b_{r,s}\sigma)$$

where

$$b_{r,s} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ p^r & & & 0 \end{bmatrix} \in \text{GL}_s(L).$$

- The $s$-tuple $(r/s, \cdots, r/s)$ is called the Newton vector of $V_\lambda$.
- Any $(V, \Phi)$ has a Newton vector $\bar{\nu}(V, \Phi) = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Q}^n_{\text{dom}}$ by decomposing $(V, \Phi)$ as a sum of simple objects and stringing together all the Newton vectors of the simple objects, in non-increasing order.
- Given $b \in GL(V)(L)$, define its Newton point $\bar{\nu}_b \in \mathbb{Q}^n_{\text{dom}}$ to be the Newton vector of the isocrystal $(L^n, b\sigma)$.
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Elementary computation of Newton points

- $\overline{\nu}_b$ is unchanged if $b$ is replaced with $g^{-1}b\sigma(g)$ (since isomorphism class of $(V, b\sigma)$ is unchanged).
- Therefore we can replace $b$ with an element of form $\epsilon^\lambda w$, i.e., a monomial matrix in $\text{GL}_n(L)$.
- Let $N$ be the order of the permutation matrix $w$. Then $\overline{\nu}_b$ is the unique dominant element in $\mathbb{Q}^n_{\text{dom}}$ which is some permutation of $\frac{1}{N} \sum_{i=0}^{N-1} w^i(\lambda)$.
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- For an \( O \)-lattice \( \Lambda \subset V \), define its **Hodge point** \( \mu = \mu(\Lambda) \in \mathbb{Z}^n_{\text{dom}} \) by \( \text{inv}(\Lambda, \Phi(\Lambda)) = \mu \). This makes sense even when \( \Phi(\Lambda) \not\subset \Lambda \).

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  (This holds in either function-field or p-adic context.)

- That is, The Hodge polygon lies above the Newton polygon (with same endpoints).

- Gives a necessary condition for non-emptiness of \( X_{\mu}^{GL_n}(b) \).

- Question: does the converse of Mazur’s \( \leq \) hold? That is, given \( \mu \geq \nu(V, \Phi) \), does there exist a lattice \( \Lambda \in V \) whose Hodge point is \( \mu \)?

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Newton and Hodge Polygons

Example

\[(1, 1, 0, 0, 0) \geq (\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})\]
Mazur inequality and non-emptiness in general

- For general $G$, Kottwitz defined notions of $G$-isocrystal, and also the **Newton point** $\bar{\nu}_b \in X_*(A)_{\mathbb{Q}, \text{dom}}$ for $b \in G(L)$.

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Mazur inequality and non-emptiness in general

- For general $G$, Kottwitz defined notions of $G$-isocrystal, and also the Newton point $\bar{\nu}_b \in X_*(A)_{\mathbb{Q},\text{dom}}$ for $b \in G(L)$.
- The inequality $\mu \geq \bar{\nu}_b$ now is for usual dominance order on $X_*(A)_{\mathbb{R},\text{dom}}$.

**Theorem**

$$X^G_\mu (b) \neq \emptyset \iff \mu \geq \bar{\nu}_b.$$  

- Kottwitz-Rapoport ($GL_n$ and $GSp_{2n}$ and reduced general case to problem on root systems), C. Lucarelli (split classical groups), Q. Gashi (general split groups).
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- Upshot: We know exactly when ADLVs in any affine Grassmannian are non-empty.
Application of $G$-isocrystals: moduli of abelian varieties over $\overline{k}$

- Dieudonné: every polarized $n$-dim'lk abelian variety $\mathcal{A}$ over $\overline{k}$ gives rise to a $\text{GSp}_{2n}$-isocrystal $(L^{2n}, b\sigma)$. The Newton point $\overline{\nu}_b$ is therefore an invariant of $\mathcal{A}$.

- Define the **Newton stratum** $\mathcal{S}_b$ in the moduli space of all $\mathcal{A}$ to consist of those $\mathcal{A}$ with fixed Newton point $\overline{\nu}_b$.

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What about \( \dim X_{\mu}^G(b) \)?

**Theorem (GHKR + Viehmann)**

If \( X_{\mu}^G(b) \neq \emptyset \), then

\[
\dim X_{\mu}^G(b) = \langle \rho, \mu - \nu_b \rangle - \frac{1}{2}(\text{rk}_F G - \text{rk}_F J_b).
\]

We write \( \text{def}_G(b) := \text{rk}_F G - \text{rk}_F J_b \).
Remarks

- $J_b(F) = \{ g \in G(L) \mid g^{-1}b\sigma(g) = b \}$.
- Conjectured by Rapoport, who pointed out the similarity with Chai’s conjecture.
- In particular, if $b = 1$, get $\dim X^G_{\mu}(1) = \langle \rho, \mu \rangle$ (cf. GL$_2$ example).
- After some work, Chai’s conjecture takes the form surprising form

$$\dim(S_b) = \langle \rho, \mu + \bar{\nu}_b \rangle - \frac{1}{2}(\text{rk}_FG - \text{rk}_FJ_b),$$

where $\mu = (1^n, 0^n)$, a cocharacter for GSp$_{2n}$. There is a geometric reason for this similarity.
- $X^G_{\mu}(b)$ is conjectured to be equidimensional. This is proved when $b \in A(L)$ [GHKR] and when $b$ is "basic" [Hartl-Viehmann].
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ADLVs in the affine flag variety

- Let $I \subset G(L)$ be an Iwahori subgroup, and call $G(L)/I$ the affine flag variety.

- $I\backslash G(L)/I = \tilde{W} = X_*(A) \rtimes W$.

- For $x \in \tilde{W}$ and $b \in G(L)$, define

$$X_{x}^{G}(b) = \{gI \in G(L)/I \mid g^{-1}b\sigma(g) \in IxI\}.$$ 

**Questions:** When are $X_{x}^{G}(b) \neq \emptyset$? Are they equidimensional? Is there a formula for the dimensions?

- Much less is known, but progress has been made.

- The following picture shows the dimensions of ADLVs for $G = G_2$, $b = 1$. 

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Thomas J. Haines
Survey of Affine Deligne-Lusztig Varieties
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**A Question in $\sigma$-linear algebra**

- Basic Questions about ADLVs
- Isocrystals and Mazur’s inequality
- Non-emptiness of ADLVs in the affine Grassmannian
- Dimensions of ADLVs in the affine Grassmannian
- ADLVs in the affine flag variety
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Some results

**Theorem (GHKR)**

(i) There is an algorithm, in terms of foldings in Bruhat-Tits building of $G(L)$, to compute $\dim X^G_x(b)$ for all $G, x,$ and $b$.

(ii) There is a conjectural (non-algorithmic) description of when $X^G_x(b)$ is empty, for $b$ "basic", and we can prove emptiness occurs when predicted.

(iii) There is a conjectural formula for $x$ "generic" and $b$ "basic" which is supported by computer evidence: write $x = w_2 e^\lambda w_1 w_2^{-1}$, for $w_i \in W$ and $\lambda \in X_*(A)_{\text{dom}}$. Conjecture:

\[ X_x(b) \neq \emptyset \iff w_1 \notin \bigcup_{T \subset S} W_T, \text{ in which case} \]

\[ \dim X^G_x(b) = \frac{1}{2}(\ell(x) + \ell(w_1) - \text{def}_G(b)). \]