70. (a) Prove that $(11)$ is a maximal ideal in the ring $\mathbb{Z}[i]$, using the method we presented in lecture for $3$ replaced by $11$ (or following the method on p. 149 of your text).

(b) Show that $\mathbb{Z}[i]/(11)$ is a finite field with $121$ elements.

71. (a) Consider the finite ring $M_2(\mathbb{F}_p)$, where $p$ is a prime number. Find the group of units $M_2(\mathbb{F}_p)^\times$ explicitly (i.e. characterize the matrices which are in this group).

(b) Find the orders of $M_2(\mathbb{F}_p)$ and of $M_2(\mathbb{F}_p)^\times$.

72. All ideals in this problem are in a commutative ring $R$.

(a) Prove that any intersection of ideals (even infinitely many) is an ideal.

(b) If $I, J$ are ideals of a commutative ring $R$, define $IJ$ to be the set of all finite sums of products of the form $xy$ for $x \in I$ and $y \in J$. Prove that $IJ$ is an ideal of $R$.

(c) Show that $IJ \subseteq I \cap J$.

(d) Prove that $I + J := \{x + y \mid x \in I, y \in J\}$ is an ideal.

(e) Prove that if $R$ has an identity $1$ and $I + J = R$, then $IJ = I \cap J$.

(f) Give a concrete example showing that, in general, the ideals $IJ$ and $I \cap J$ might not be equal.

73. Using induction, generalize the Chinese Remainder Theorem to the following statement: if $m_1, m_2, \ldots, m_t$ are pairwise coprime positive integers, and if $a_1, a_2, \ldots, a_t$ are any integers, then there exists $x \in \mathbb{Z}$ such that $x \equiv a_i \pmod{m_i}$ for all $i$, and such an $x$ is uniquely determined modulo $m_1 m_2 \cdots m_t$ by these conditions.