21. (a) Let $G$ be a finite group. Show that there exists a positive integer $n$ such that $a^n = e$ for all $a \in G$. (We call the smallest such $n$ the exponent of $G$.)

(b) Find the exponent of $S_3$.

22. Let $D_n = \langle a, b \mid a^2 = e, b^n = e, aba^{-1} = b^{-1} \rangle$ denote the dihedral group of order $2n$. Recall every element has a unique expression of the form $a^i b^j$ for $i \in \{0, 1\}$ and $j \in \{0, 1, \ldots, n-1\}$. Find, in terms of these expressions, exactly which elements are reflections about a line through the origin. Start with the cases $n = 4$ and $n = 5$ and draw pictures of all the lines fixed by such reflections in each of those cases.

23. (a) Let $F = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. Show by direct computation that matrix multiplication in $M_2(F)$ and $M_3(F)$ is associative.

(b) Use associativity of linear transformations of $F^n$ (the vector space of column vectors) to give a conceptual proof of the associativity of matrix multiplication in $M_n(F)$.

24. Determine whether each matrix has an inverse, and if so find it.

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

(c) $C = AB$.

25. Consider the set of strictly upper triangular $3 \times 3$ matrices $U_3(F)$, that is, matrices of the form

$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$

for $a, b, c \in F$. Show that $U_3(F)$ is a group under matrix multiplication, and that it is nonabelian.

26. Suppose that $G$ is a group.

(a) Show that $G$ is abelian if and only if $(ab)^2 = a^2 b^2$ for all $a, b \in G$.

(b) Show that if $a^2 = e$ for all $a \in G$, then $G$ is abelian.

(c) Show that if $(ab)^i = a^i b^i$ for three consecutive integers $i$, then $G$ is abelian.