15. (5 points) Suppose \( n \geq 3 \). Show that \( Z(S_n) = 1 \).

16. (10 points) Suppose \( G \) is a group with \( |G| > 2 \) (the group need not be finite). Show that \( |\text{Aut}(G)| > 1 \). You may assume that every vector space has a basis.

17. Prove the following assertions (the first two we made in lecture).
   (a) (5 points) The only elements in \( S_5 \) which commute with \( \pi := (1\ 2\ 3\ 4\ 5) \) are even. In fact, show the centralizer of \( \pi \) is simply \( \langle \pi \rangle \).
   (b) (5 points) The elements \( (1\ 2\ 3\ 4\ 5) \) and \( (1\ 3\ 5\ 2\ 4) \) are conjugate in \( S_5 \) but not in \( A_5 \).
   (c) (5 points) Show that \( (1\ 2\ 3\ 4\ 5) \) and \( (1\ 3\ 5\ 2\ 4) \) are conjugate as members of \( A_7 \) but not as members of \( A_6 \).

18. (10 points) Show that a \( d \)-cycle in \( S_n \) cannot be written as a product of fewer than \( d - 1 \) transpositions. HINT: Say \( \gamma = \sigma_1 \cdots \sigma_k \), where \( \sigma_i \) is a transposition. Consider \( P(\gamma) = P(\sigma_1) \cdots P(\sigma_k) \) in \( \text{GL}_n(\mathbb{C}) \). Let \( H_i = \ker(P(\sigma_i) - 1) \) and \( K = \ker(P(\gamma) - 1) \). Show that \( K \supseteq \cap_i H_i \). On the other hand, show that \( \dim(\cap_i H_i) \geq n - k \) and \( \dim(K) = (n - d) + 1 \).

19. (10 points) Let \( G \) be a finite group acting on a finite set \( X \). For \( g \in G \), let \( f(g) \) denote the number of elements in \( X \) which are fixed by \( g \). Prove Burnside’s formula

\[
\frac{1}{|G|} \sum_{g \in G} f(g) = \text{number of orbits}.
\]