52. Dummit-Foote, 10.3, #16-18.

53. (5 points) Suppose \( f : V \to V \) is a homomorphism of the \( R \)-module \( V \) to itself. Suppose that \( f^2 = f \). Prove that \( V \cong \ker(f) \oplus \operatorname{im}(f) \) as \( R \)-modules.

54. (a) (5 points) (Cramer’s Rule). Here \( R \) is any commutative ring with 1. Let \( X \) be any \( n \times n \) matrix over \( R \). Define the adjoint matrix \( \operatorname{adj}(X) \) by

\[
\operatorname{adj}(X)_{ij} = (-1)^{i+j} \det(X_{ji}),
\]

where \( X_{ji} \in M_{n-1}(R) \) is \( X \) without its \( j \)th row and \( i \)th column. Prove Cramer’s Rule

\[
X \cdot \operatorname{adj}(X) = \operatorname{adj}(X) \cdot X = \det(X) I_n.
\]

HINT: You may assume the result for \( R \) any field. The identity is equivalent to \( 2n^2 \) polynomial relations in the entries of \( X \). It is enough to prove these relations hold in the polynomial ring \( \mathbb{Z}[X_{ij}] \) in \( n^2 \) variables \( X_{ij} \). Explain why and show that these follow from Cramer’s rule for the field \( \mathbb{Q}(X_{ij}) = \operatorname{Frac}(\mathbb{Z}[X_{ij}]) \).

(b) (5 points) Similarly, show that \( \det(XY) = \det(X) \det(Y) \) for any \( X, Y \in M_n(R) \), assuming this fact holds for \( R \) any field.

(c) (5 points) Deduce that \( A \in M_n(R) \) has a multiplicative inverse in \( M_n(R) \) if and only if \( \det(A) \in R^* \).

55. Here \( R \) is any commutative ring with 1.

(a) (10 points) If \( A \in M_d(R) \) has \( \det(A) = 0 \), then there exists \( x \neq 0 \) in \( R^d \) with \( Ax = 0 \). HINT: WLOG \( A \) is not the zero matrix. Consider the largest non-vanishing minor in \( A \) and use Cramer’s rule.

(b) (5 points) If \( V \) is any \( R \)-module generated by \( n \) elements \( x_1, \ldots, x_n \), and \( y_1, \ldots, y_d \) is any set of \( d \) elements in \( V \) which are linearly independent over \( R \), then \( d \leq n \).

56. Let \( R \) be a Noetherian ring (as usual, commutative with 1). Let \( M \) be any \( R \)-module with \( M \neq 0 \). Show that there exists \( x \in M \) with \( \operatorname{Ann}(x) \) a prime ideal in \( R \).