Solutions to Homework 10

52. (Dummit-Foote 10.3 #16) The kernel of the map $M \to M/A_1 M \times \cdots \times M/A_k M$ is the set of all $m \in M$ such that $m \in A_i M$ for all $i$. But this is exactly $A_1 M \cap \cdots \cap A_k M$.

(Dummit-Foote 10.3 #17) The same proof that works for the ring $R$ in the text works for a module $M$ just by making the obvious substitutions. Here is another proof, using tensor products.

Since the ideals $A_i$ are all comaximal, we have by the Chinese Remainder Theorem for the ring $R$

$$R/A_1 \cdots A_k \simeq R/A_1 \times \cdots \times R/A_k.$$  \hspace{1cm} (1)

Now for any ideal $I$ of $R$ there is an isomorphism $(R/I) \otimes_R M \simeq M/IM$ of $R$-modules. As tensor products commute with finite direct products, we can apply $- \otimes_R M$ to the isomorphism (1), to obtain an isomorphism

$$M/A_1 \cdots A_k M \simeq M/A_1 M \times \cdots \times M/A_k M$$

of $R$-modules.

(Dummit-Foote 10.3 #18) As $(a)M = 0$, we have $M/(a)M = M$. The ideals $p_i^{\alpha_i}$ are all comaximal, so we can apply the Chinese Remainder Theorem to obtain an isomorphism

$$M \simeq M/(p_1^{\alpha_1})M \times \cdots \times M/(p_k^{\alpha_k})M.$$ 

Now let $m = (m_1, \ldots, m_k)$ be an element of the right hand side, and suppose that $p_i^{\alpha_i}m = 0$. Then $p_i^{\alpha_i}m_j = 0$ for all $j$. If $j \neq i$, then $p_i^{\alpha_i}m_j = 0$ as well, and since $(p_1^{\alpha_1})$ and $(p_j^{\alpha_j})$ are comaximal, this implies that $m_j = 0$. Thus the $p_i^{\alpha_i}$-primary component of the right hand side is $M/(p_i^{\alpha_i}M)$.

The isomorphism obtained from the Chinese Remainder Theorem thus gives a direct sum (= direct product, since the number of factors is finite) decomposition of $M$ into its $p_i^{\alpha_i}$-primary components. But by definition, the $p_i^{\alpha_i}$-primary component of $M$ is $M_i$. Thus

$$M = M_1 \oplus \cdots \oplus M_k.$$ 

53. We have an exact sequence

$$0 \to \ker(f) \to V \xrightarrow{f} \text{im}(f) \to 0$$

of $R$-modules. The map $f : \text{im}(f) \to V$ defines a section of $f : V \to \text{im}(f)$. To see this, note that if $f(x) \in \text{im}(f)$, then $f(f(f(x))) = f(f(x)) = f(x)$, because $f^2 = f$. Thus the exact sequence splits.

54. (a) Consider first an $n \times n$ matrix $X = (X_{ij})$ of variables $X_{ij}$, where $X_{ij} \in \mathbb{Z}[X_{ij}]$. We must prove the relations $X \text{adj}(X) = \det(X)I_n$ and $\text{adj}(X)X = \det(X)I_n$. Now, $\det(X)$ and the entries of $\text{adj}(X)X$ and $X \text{adj}(X)$ are all polynomials in the $X_{ij}$, with coefficients only $\pm 1$, thus they live in $\mathbb{Z}[X_{ij}]$. However, this polynomial ring embeds in its field of fractions, where we are assuming Cramer’s Rule to hold. Hence the polynomial relations hold for an $n \times n$ matrix of indeterminates.

Now if $X = (x_{ij})$ is an $n \times n$ matrix with entries in $R$, simply substitute $x_{ij}$ for $X_{ij}$ in the equations we obtained from $X \cdot \text{adj}(X) = \det(X)I_n$ and $\text{adj}(X)X = \det(X)I_n$. This proves Cramer’s
Rule for $R$.

(b) This is a similar argument to part (a). First consider matrices $X = (X_{ij})$ and $Y = (Y_{ij})$. Proving the equation $\det(XY) = \det(X)\det(Y)$ amounts to proving that $n^2$ equations in the polynomial ring $\mathbb{Z}[X_{ij}, Y_{ij}]$ are true. But this polynomial ring embeds in $\mathbb{Q}(X_{ij}, Y_{ij})$, so the result holds for matrices of indeterminates.

Now if $X = (x_{ij})$ and $Y = (y_{ij})$ are elements of $M_n(R)$, substitute the $x_{ij}$ for the $X_{ij}$, and the $y_{ij}$ for the $Y_{ij}$, into the equations proving the result for $\mathbb{Z}[X_{ij}, Y_{ij}]$. This yields the formula in question for arbitrary $R$.

(c) Suppose that $A \in M_n(R)$ has an inverse $B$, so that $AB = I$. Then $1 = \det(AB) = \det(A)\det(B)$ by part (b), so $\det(A) \in R^\times$.

Conversely, if $\det(A) \in R^\times$, then by Cramer’s Rule we have $A \cdot (\det(A)^{-1} \text{adj}(A)) = I$.

56. Let $S = \{ \text{Ann}(x) \mid x \in M \}$. Then $S$ is non-empty, because if $x \in M$ is non-zero, then $0 \in \text{Ann}(x)$ and $1 \not\in \text{Ann}(x)$, so $\text{Ann}(x)$ is a non-empty proper ideal of $R$. By the Noetherian hypothesis, $S$ has a maximal element, call it $\text{Ann}(x_0)$. We will prove that $\text{Ann}(x_0)$ is prime.

Let $a, b \in R$, and suppose that $ab \in \text{Ann}(x_0)$ and $b \not\in \text{Ann}(x_0)$ (so that $bx_0 \neq 0$). Then $abx_0 = 0$, so $a \in \text{Ann}(bx_0)$. But anything that annihilates $x_0$ annihilates $bx_0$, hence $\text{Ann}(x_0) \subseteq \text{Ann}(bx_0)$. By maximality, $\text{Ann}(x_0) = \text{Ann}(bx_0)$, so $a \in \text{Ann}(x_0)$, and it follows that the ideal is prime.