61. Let $F$ be any field, and $E \supseteq F$ any extension field. Let $A, B \in M_n(F)$. Suppose that $A$ and $B$ are similar when viewed as elements of $M_n(E)$. Show that $A$ and $B$ are similar as elements of $M_n(F)$.

62. Let $F$ be any field, and $A, B \in M_n(F)$. Show that $A$ and $B$ are similar if and only if the matrices $XI - A$ and $XI - B$ have the same Smith forms in $M_n(F[X])$.

63. Prove that any matrix $A \in M_n(F)$ is similar to its transpose $A^t \in M_n(F)$.

64. (a) Let $A \in M_n(F)$. Let $\overline{F}$ be an algebraic closure of $F$ (you may assume this exists). Show that the minimal polynomial $\min(A)$ and the characteristic polynomial $\text{char}(A)$ have the same roots in $\overline{F}$ (neglecting multiplicities).

(b) Suppose that $A$ is a $2 \times 2$ or a $3 \times 3$ matrix over a field $F$. Show that the invariant factors of $A$ (hence the rational canonical form of $A$) can be computed once one knows $\text{char}(A)$ and $\min(A)$.

65. We say $A \in M_n(F)$ is diagonalizable over $F$ if there exists $P \in \text{GL}_n(F)$ such that $P^{-1}AP$ is a diagonal matrix.

(a) Let $\overline{F}$ denote an algebraic closure of $F$. Show that $A \in M_n(\overline{F})$ is diagonalizable over $\overline{F}$ if and only if its Jordan form in $M_n(\overline{F})$ is diagonal.

(b) Let $A \in M_n(F)$. Show that $A$ is diagonalizable over $F$ if and only $\min(A)$ factors in $F[X]$ as a product of pairwise distinct linear factors.

66. Use the “minors algorithm” discussed in class to find the invariant factors and the RCF for the matrix

$$
\begin{bmatrix}
-1 & -2 & 6 \\
-1 & 0 & 3 \\
-1 & -1 & 4
\end{bmatrix}
$$

(To check your answer compare it with what you can derive just using #64 (b) above.)