61. Suppose that $A, B \in M_n(F)$ are similar over $E$, and let $M_A$ and $M_B$ be their rational canonical forms computed over $F$. Then $M_A$ and $M_B$ satisfy the definition of rational canonical form when viewed as matrices with entries in $E$. But over $E$, $A$ and $B$ are similar, thus they have the same rational canonical form. Hence $M_A = M_B$, and it follows that $A$ and $B$ are similar over $F$.

62. Recall that for a matrix $A \in M_n(F)$, the Smith Normal Form of $XI - A$ is a diagonal matrix with $(1, \ldots, 1, a_1(x), \ldots, a_m(x))$ along the diagonal, where $a_1(x)| \cdots |a_m(x)$ are the invariant factors of $A$. Obviously then, $XI - A$ and $XI - B$ have the same Smith Normal Form if and only if $A$ and $B$ have the same invariant factors. But $A$ and $B$ have the same invariant factors if and only if they have the same rational canonical form.

63. By the previous problem, we must show that $XI - A$ and $XI - A^t = (XI - A)^t$ have the same Smith form. Whatever row and column operations must be done to get $XI - A$ in its Smith form, we can perform the “transpose” operations to $(XI - A)^t$, and the result will be the transpose of the Smith form of $XI - A$. But the Smith form of $XI - A$ is diagonal and hence equal to its own transpose. Thus $XI - A$ and $(XI - A)^t$ have the same Smith form, which shows that $A$ and $A^t$ are similar.

64. (a) Let $a_1(x)| \cdots |a_m(x)$ be the invariant factors of $A$. Then $\text{min}(A) = a_m(x)$ and $\text{char}(A) = \prod a_i(x)$. Thus the minimal polynomial divides the characteristic polynomial, so every root of the minimal polynomial is a root of the characteristic polynomial. Conversely, every root of every $a_i$ is a root of the minimal polynomial by the divisibility condition. Hence every root of the characteristic polynomial is a root of the minimal polynomial, therefore they have the same roots.

(b) First note that for any $n \times n$ matrix $A$ with invariant factors $a_1(x)| \cdots |a_m(x)$, we have $\text{char}(A) = \prod a_i(x)$, and $\deg(\text{char}(A)) = n$. Thus $\sum \deg(a_i(x)) = n$.

Suppose that $A$ is $2 \times 2$. The characteristic polynomial $c(x)$ of $A$ is degree two. If the minimal polynomial $m(x)$ is degree one, then the invariant factors of $A$ are $m(x), m(x)$ and $m(x)^2 = c(x)$.

Now we suppose $A$ is $3 \times 3$, then $c(x)$ is degree three. If $m(x)$ is degree one, then the invariant factors are $m(x), m(x), m(x)$ and $m(x)^3 = c(x)$. If $m(x)$ is degree two, then the invariant factors are $c(x)/m(x), m(x)$. If $m(x)$ is degree three, then it is the only invariant factor, and $m(x) = c(x)$.

65. (a) Suppose that the Jordan form of $A$ is diagonal. Since every matrix is similar to its Jordan form, this shows that $A$ is diagonalizable. Conversely, if $A$ is diagonalizable such that $P^{-1}AP$ is diagonal, then $P^{-1}AP$ is the Jordan form of $A$ by uniqueness of the Jordan form. So the Jordan form of $A$ is diagonal.

(b) Suppose that $A$ is diagonalizable. Since $\text{min}(A)$ depends only on the similarity class of $A$, we may assume $A$ is diagonal. The minimal polynomial of a diagonal matrix with $(a_1, \ldots, a_n)$ down
the diagonal is

\[ m(X) = \prod_{\text{distinct } a_i} (X - a_i), \]

which proves the assertion.

Conversely, suppose that \( a_1(x) | \cdots | a_m(x) \) are the invariant factors of \( A \). Since \( a_m(x) \) splits as a product of distinct linear factors, so must all of the other \( a_i(x) \)'s. We know that

\[ V = F[x]/(a_1(x)) \oplus \cdots \oplus F[x]/(a_m(x)) \]

so by the Chinese Remainder Theorem, we have that

\[ V = \bigoplus_{\alpha} F[x]/(x - \alpha) \]

for some \( \alpha \)'s. But this says that the matrix \( A \) acts on \( V \) as a direct sum of \( 1 \times 1 \) matrices, that is, diagonally.

66. The rational canonical form of

\[
\begin{pmatrix}
-1 & -2 & 6 \\
-1 & 0 & 3 \\
-1 & -1 & 4
\end{pmatrix}
\]

is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 2
\end{pmatrix}.
\]

The invariant factors are \( x - 1, (x - 1)^2 \).