12. (a) (5 points) Show that any group of order 33 is cyclic.

(b) (5 points) Show that any group of order $35^2$ is abelian. Up to isomorphism, how many groups of order $35^2$ are there?

13. (5 points) Let $G$ be a $p$-group. Show that $G$ possesses a chain of subgroups $1 < H_1 < \cdots < H_n = G$, each normal in its successor, such that each quotient group $H_i/H_{i-1}$ is cyclic.

14. Let $p$ denote a prime number and let $\mathbb{F}_p$ denote the finite field of cardinality $p$. Let $G = \text{GL}_n(\mathbb{F}_p)$.

(a) (5 points) Show that $|G| = (p^n - 1)(p^n - p)\cdots(p^n - p^{n-1})$. HINT: Let $g \in \text{GL}_n(\mathbb{F}_p)$. The first column of $g$ may be any non-zero vector. The second column may be any vector not in the $\mathbb{F}_p$-span of the first column. Etc.

(b) (5 points) Find a $p$-Sylow subgroup of $G$. Hint: For $n = 2$ i.e. for $G = \text{GL}_2(\mathbb{F}_p)$, the group $U$ of elements of the form $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ for $x \in \mathbb{F}_p$ is a $p$-Sylow subgroup. Check this and try to generalize to general $n$.

(c) (5 points) For $n = 2$, show that the normalizer of $U$ above contains $B$, the subgroup consisting of the upper triangular matrices in $\text{GL}_2(\mathbb{F}_p)$.

(d) (5 points) For $n = 2$, show that the number of $p$-Sylow subgroups in $G$ is $1 + p$ and simultaneously show that the normalizer of $U$ above is precisely the subgroup $B$.

(e) (BONUS 10 points extra credit) For $n$ general, find the normalizer of your $p$-Sylow subgroup from part (b), and use this information to prove that the number of $p$-Sylow subgroups in $G$ is

$$\frac{p^n - 1}{p - 1} \cdot \frac{p^{n-1} - 1}{p - 1} \cdots \frac{p^2 - 1}{p - 1} \cdot \frac{p - 1}{p - 1}.$$

(Or, you are welcome to find another way to compute the number of $p$-Sylow subgroups.)

15. Prove the following assertions (the first two we made in lecture).

(a) (5 points) The only elements in $S_5$ which commute with $\pi := (1\ 2\ 3\ 4\ 5)$ are even. In fact, show the centralizer of $\pi$ is simply $\langle \pi \rangle$.

(b) (5 points) The elements $(1\ 2\ 3\ 4\ 5)$ and $(1\ 3\ 5\ 2\ 4)$ are conjugate in $S_5$ but not in $A_5$.

(c) (5 points) Show that $(1\ 2\ 3\ 4\ 5)$ and $(1\ 3\ 5\ 2\ 4)$ are conjugate as members of $A_7$ but not as members of $A_6$. 
16. (10 points) Show that a $d$-cycle in $S_n$ cannot be written as a product of fewer than $d-1$ transpositions. HINT: Say $\gamma = \sigma_1 \cdots \sigma_k$, where $\sigma_i$ is a transposition. Consider $P(\gamma) = P(\sigma_1) \cdots P(\sigma_k)$ in $\text{GL}_n(\mathbb{C})$. Let $H_i = \ker(P(\sigma_i) - 1)$ and $K = \ker(P(\gamma) - 1)$. Show that $K \supseteq \cap_i H_i$. On the other hand, show that $\dim(\cap_i H_i) \geq n - k$ and $\dim(K) = (n - d) + 1$. 