17. (10 points) Let $G$ be a finite group acting on a finite set $X$. For $g \in G$, let $f(g)$ denote the number of elements in $X$ which are fixed by $g$. Prove Burnside’s formula

$$\frac{1}{|G|} \sum_{g \in G} f(g) = \text{number of orbits}.$$ 

HINT: Count the set $\{(g, x) \mid gx = x\}$ in two different ways.

18. (5 points) Suppose $n \geq 5$. Show that the only normal subgroups of $S_n$ are 1, $A_n$, and $S_n$. You may use the fact, proved in class, that $A_n$ is simple.

19. (a) (5 points) Let $G$ be a $p$-group, $G \neq 1$. Let $N \trianglelefteq G$, $N \neq 1$. Show that $N \cap Z(G) \neq 1$.

(b) (5 points) Let $G$ be a non-abelian group of order $p^3$ (where $p$ is prime). Show that $|Z(G)| = p$ and that $Z(G)$ has no complement in $G$.

20. (10 points) Show that a group of order $2^2 \cdot 5 \cdot 19$ is not simple.

21. (15 points) Let $G$ be a finite group of order $p^bn$ (here we do not assume $(p, n) = 1$). Let $H$ be a subgroup of order $p^a$, where $0 \leq a \leq b$. Let $N(p^b, H)$ denote the number of subgroups of $G$ which contain $H$ and have order $p^b$. The following steps will prove that $N(p^b, H) \equiv 1 \mod p$. (This gives another proof of a (stronger) version of Sylow’s theorems.)

(a) Let $\Omega$ denote the set of subsets of $G$ which have order $p^b$ and are stable under left multiplication by elements of $H$. The group $G$ acts on $\Omega$ by $M \mapsto Mg$. Let $\{T_i\}_{i \in I}$ denote the orbits. For $M_i \in T_i$, let $G_i$ denote the stabilizer of $M_i$. Show that $|G_i|$ is a divisor of $p^b$.

(b) Show that $|T_i| = n \Leftrightarrow |G_i| = p^b$. Show also that $|T_i| \neq n \Leftrightarrow |G_i| < p^b$, in which case $|T_i| \equiv 0 \mod p$.

(c) Prove that there is a 1-1 correspondence between orbits $T_i$ having length exactly $n$ and subgroups $U$ of $G$ which contain $H$ and have order exactly $p^b$.

(d) Deduce that

$$|\Omega| \equiv \sum_{|T_i| = n} |T_i| \equiv nN(p^b, H) \mod p.$$ 

(e) Show that $|\Omega| = \binom{p^b-a}{p^b-a} \cdot nN(p^b, H) \mod p$. 

$$\binom{p^b-a}{p^b-a} \equiv nN(p^b, H) \mod p.$$
(f) By considering the above equation in the case of a cyclic group of order $p^n$, show that

$$\binom{p^n - a}{p^n} \equiv n \pmod{p}.$$ 

(g) Conclude that $N(p^n, H) \equiv 1 \pmod{p}$.

22. (10 points) Let $P$ be a $p$-Sylow subgroup of a finite group $G$. Show that $N_G(N_G(P)) = N_G(P)$. 