Solutions to Homework 5

23. Suppose $B$ is finite. Then as $A$ injects into $B$, and $B$ surjects onto $C$, $A$ and $C$ must both be finite as well. Conversely, suppose that $A$ and $C$ are both finite. We can write $B$ as a union of $|C|$ cosets of $A$. So $B$ is a finite union of finite sets and thus finite.

If $A$, $B$ and $C$ are all finite, then the first isomorphism theorem yields $B/A \cong C$. Thus $|B|/|A| = |C|$, so $|B| = |A| \cdot |C|$.

24. Let $n_{17}$ be the number of Sylow 17-subgroups of $G$. Sylow’s theorem tells us that $n_{17} \equiv 1 \mod 17$ and $n_{17} | 55$, thus $n_{17} = 1$. Let $P$ denote the Sylow 17-subgroup, then $P \trianglelefteq G$. By assumption, $G$ has an element of order 55, call it $g$ and let $C = \langle g \rangle$. The orders of $P$ and $C$ are relatively prime, thus $P \cap C = 1$, and as $P$ is normal in $G$, we have $G = P \rtimes C$.

Let $\phi : C \to \text{Aut}(P)$ be the homomorphism giving the semi-direct product. As $\text{Aut}(P)$ has order 16, we have that the image of $\phi$ must have order dividing 16 and 55, thus $\phi$ is trivial. Hence $G = P \times C \cong \mathbb{Z}/17 \mathbb{Z} \times \mathbb{Z}/55 \mathbb{Z} \cong \mathbb{Z}/(17 \cdot 55) \mathbb{Z}$.

25. Let $g \in G$ be the element of $G$ such that the action of 1 on $G$ is conjugation by $g$. Note that in $G \rtimes \mathbb{Z}$, we have

$$(x, n)(y, m) = (x \cdot n \cdot y, n + m) = (x g^n y g^{-n}, n + m)$$

Define a map $\phi : G \rtimes \mathbb{Z} \to G \times \mathbb{Z}$ by $\phi(x, n) = (x g^n, n)$. This is a homomorphism because

$$\phi((x, n)(y, m)) = \phi(x g^n y g^{-n}, n + m)$$

$$= (x g^n y g^m, n + m)$$

$$= (x g^n, n)(y g^m, m)$$

$$= \phi(x, n)\phi(y, m).$$

The map $\phi$ is injective because $(x g^n, n) = (e, 0)$ implies $n = 0$ and $x = e$. The map $\phi$ is surjective because for any $(x, n) \in G \times \mathbb{Z}$, $\phi(x g^{-n}, n) = (x, n)$. Thus $\phi$ is an isomorphism.

26. Let $G = H \rtimes \phi \text{Aut}(H)$, where $\phi : \text{Aut}(H) \to \text{Aut}(H)$ is the identity map. Then $H$ is a subgroup of $G$ as the set of all elements of the form $(h, 1)$. For $\sigma \in \text{Aut}(H)$, consider the element $(1, \sigma) \in G$. Conjugation by $(1, \sigma)$ on any $(h, 1)$ gives

$$(1, \sigma)(h, 1)(1, \sigma^{-1}) = (\sigma(h), \sigma(1), \sigma^{-1})$$

$$= (\sigma h, \sigma(1), \sigma^{-1})$$

$$= (\sigma h, 1).$$

Thus conjugation by $(1, \sigma)$ restricted to $H$ is simply $\sigma$, as desired.

27. (Dummit-Foote, 6.1 #12) For any group $G$ where $Z(G) = 1$, the Upper Central Series of $G$ has $Z_i(G) = 1$ for all $i$. Since $Z(A_4) = 1 = Z(S_4)$, we have trivial Upper Central Series for both of these groups.
To compute the Lower Central Series of $A_4$, note that the only non-trivial normal subgroup of $A_4$, $K = \{1, (12)(34), (13)(24), (14)(23)\}$, has index 3 and thus the quotient $A_4/K$ is abelian. Hence $[A_4, A_4] = K$. To compute $[A_4, K]$, note that

$$(a \ b \ c)(a \ b)(c \ d)(a \ c \ b)(a \ b)(c \ d) = (a \ d)(b \ c)$$

hence every product of disjoint two cycles is in $[A_4, K]$, thus $[A_4, K] = K$. It follows that

$$A_4 \geq K \geq K \geq \ldots$$

is the Lower Central Series for $A_4$.

To compute the Lower Central Series of $S_4$, note that the commutator of the elements $(a \ b)$ and $(a \ b \ c)$ is $(a \ b \ c)$. As $A_4$ is generated by 3-cycles, we have $A_4 \leq [S_4, S_4]$. Since the quotient $S_4/A_4$ is abelian, we must have $[S_4, S_4] = A_4$. Now $(a \ b \ c) \in A_4$, so the above commutator is actually an element of $[S_4, A_4]$. Thus $A_4 = [S_4, A_4]$, and it follows easily that

$$S_4 \geq A_4 \geq A_4 \geq \ldots$$

is the Lower Central Series for $S_4$.

(Dummit-Foote, 6.1 #13) Since $S_n$ and $A_n$ have trivial centers for $n \geq 5$, they have $Z_i = 1$ for all $i$, as above.

Now $A_n$ is simple for $n \geq 5$, hence $[A_n, A_n] = 1$ or $[A_n, A_n] = A_n$. As $A_n$ is non-abelian, we must have $[A_n, A_n] = A_n$, hence the Lower Central Series of $A_n$ is

$$A_n = A_n = \ldots$$

For $n \geq 5$, the only non-trivial normal subgroup of $S_n$ is $A_n$, and the quotient is order two and hence abelian. Thus $[S_n, S_n] = A_n$. Now $[S_n, A_n]$ must contain $[A_n, A_n] = A_n$, hence $[S_n, A_n] = A_n$. Thus the Lower Central Series for $S_n$ is

$$S_n \geq A_n \geq A_n \geq \ldots$$