59) (5 points). Let $\phi : V \to V$ and $\psi : W \to W$ be $F$-linear endomorphisms of finite-dimensional $F$-vector spaces $V$ and $W$. Show that $\text{Tr}(\phi \otimes \psi) = \text{Tr}(\phi) \cdot \text{Tr}(\psi)$.

Let $A$ and $B$ be $m \times m$ and $n \times n$ matrices representing the endomorphisms $\phi$ and $\psi$ respectively, in some choice of bases for $V \simeq F^m$ and $W \simeq F^n$. The matrix of $\phi \otimes \psi$ is the Kronecker product $A \otimes B$ of size $mn \times mn$ consisting of $n \times n$ blocks $a_{ij}B$. Thus $\text{Tr}(\phi \otimes \psi) = \text{Tr}(A) \cdot \text{Tr}(B) = \text{Tr}(\phi) \cdot \text{Tr}(\psi)$.

60) (10 points). Let $F$ be a field. We say $u \in GL_n(F)$ is unipotent if $u - 1$ is nilpotent, that is, $(u - 1)^N = 0$ for some $N \geq 1$.

a) Show that if $\text{char}(F) = 0$, then the only finite-order unipotent element is the identity matrix.

The minimal polynomial of a nilpotent matrix $\eta$ is $X^k$, where $\eta$ is nilpotent of order $k$. The characteristic polynomial is $X^n$. Thus the $F[X]$-module $F^n$ is of the form $F[X]/(X^{k_1}) \oplus \cdots \oplus F[X]/(X^{k_l})$ with $\max\{k_i \mid 1 \leq i \leq l\} = k$. Thus there is a basis for $F^n$ such that the matrix $\eta$ can be chosen to be block diagonal with $l$ blocks $\eta_i$, which are standard nilpotent matrices of size $k_i \times k_i$ (with 1’s on the sub-diagonal and 0’s elsewhere). Let $\eta := u - 1$. Let $u^n = 1$ where $n$ is a multiple of the order of $u$ such that $n \geq k$. Then $1 = u^n = \sum_{i=0}^{k-1} \binom{n}{i} \eta^i$. If $k = 1$, then $\eta = 0$ and $u = 1$, hence we are done. If $k > 1$, then the linear independence of the matrices $\eta^i$ for $1 \leq i \leq k - 1$ implies $\binom{n}{i} = 0$, which is possible only in characteristic $p$. Thus, if $\text{char}(F) = 0$, then $k = 1$ and $u = 1$.

b) Use (a) to prove that if $\rho$ is a representation of a finite group $G$ over $\mathbb{C}$, then $\rho(g)$ is similar to a diagonal matrix. HINT: use the Jordan-normal form of $\rho(g)$.

We choose a basis of $\mathbb{C}^n$ such that the matrix $\rho(g)$ (for a particular $g \in G$, that is) is in Jordan canonical form. Thus, we have $\mathbb{C}^n = \oplus_{i=1}^l V_i$ with $\rho(g)_i : V_i \to V_i$ given by $\rho(g)_i = \lambda_i I + \eta_i$, where $I$ is the identity matrix and $\eta_i$ is a (standard) nilpotent matrix (of size $\dim(V_i) \times \dim(V_i)$). Since $\rho(g)$ is nonsingular, we have $\lambda_i \neq 0$ (in fact $\lambda_i$ is a root of unity), therefore $\rho(g)_i/\lambda_i : V_i \to V_i$ is unipotent and by part a) $\rho(g)_i = \lambda_i I$. Thus $\rho(g)$ is diagonal.

c) Show that the conclusion of (a) is false if $\text{char}(F) = p$. 

We take $u \in GL_2(\mathbb{F}_p)$ to be the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Clearly $u^p = 1$ and $(u-1)^2 = 0$, so that $u$ is a finite order unipotent matrix with $u \neq 1$.

61) [D-F] 18.1 #16 (5 points). One-dim’l $\mathbb{C}$-reps. of a finite abelian group $G$.

Since conjugation in $GL_1(\mathbb{C}) = \mathbb{C}^\times$ is trivial, distinct 1-dim’l representations of any group $G$ are inequivalent. We can multiply two 1-dim’l reps. $\rho, \rho'$ by the rule $(\rho \rho')(g) = \rho(g)\rho'(g)$ because $\mathbb{C}^\times$ is abelian. This gives the set of inequivalent 1-dim’l reps of $G$. We can multiply two 1-dim’l reps.

Next if $G$ is a finite abelian group, we can write $G = \oplus_{i=1}^t \mathbb{Z}/n_i \mathbb{Z}$ with $n_i | n_{i+1}$. Any representation of $\rho : G \to \mathbb{C}^\times$ takes the form $\rho(g_1, \cdots, g_t) = \prod_{i=1}^t \rho_{k_i n_i}(g_i)$. Thus $\hat{G} = G$, again.

62) [D-F] 18.3 #6-8 (30 points).

6): Let $\chi_1, \cdots, \chi_r$ be the irreps. of $G$, where $r$ is the number of conjugacy classes of $G$. Let $\psi$ be the character of the given representation $\phi : G \to GL(V)$, then we know $\psi = \sum_{i=1}^r a_i \chi_i$, for some non-negative integers $a_i$, where $a_i$ is the number of copies of the irrep. corresponding to $\chi_i$ in $\phi$. Since the $\chi_i$ are orthonormal, we have $(\psi, \chi_1) = a_1 = \# \text{copies of the trivial rep. in } \phi$. The latter quantity is the dimension of the subspace $W = \{ v \in V | \phi(g)v = v \text{ for all } g \in G \}$.

7 a): If $V = \bigoplus_{i=1}^n \mathbb{C}e_i$, with $G$ permuting the basis vectors $\mathcal{B} = \{ e_i | i = 1 \cdots n \}$. Let $\mathcal{B}_i$ for $i = 1 \cdots t$ be the orbits of the $G$-action on $\mathcal{B}$. We have $V = \bigoplus_{i=1}^t V_i$ as $CG$-modules, where $V_i$ is the $\mathbb{C}$-span of $\mathcal{B}_i$.

7 b): If $W = \mathbb{C}(\sum_{e_j \in \mathcal{B}_i} a_j e_j)$ is a 1-dim’l subspace of $V_i$ which is $G$-invariant then clearly the $a_i$ have to be identical, so that $W = \mathbb{C}(\sum_{e_j \in \mathcal{B}_i} e_j)$. Thus the number of copies of the trivial rep. in the representation of $G$ on $V_i$ is exactly one.

7 c): Since, the number of copies of the trivial rep. in the representation of $G$ on $V_i$ is exactly one, it follows that $t = \# \text{G-orbits of } \mathcal{B} = \# \text{ copies of the trivial rep. in the representation of } G \text{ on } V$.

8): Since $G < S_n$, the $\mathbb{C}$-span, $V$, of $\mathcal{B} = \{ 1, 2, \cdots, n \}$ is a $CG$-module (permuting the basis $\mathcal{B}$). For $g \in G$, the number of basis elements fixed by $g$, is $\text{Fix}(g) = \psi(g)$, where $\psi$ is the character of the representation $V$. Thus $\sum_{g \in G} \text{Fix}(g) = |G|(\psi, \chi_1)$ where $\chi_1$ is the character of the trivial rep. By Problem 6) above, $(\psi, \chi_1) = \# \text{ copies of the trivial rep. in } V$, and by Problems 7 a) and b) above, this quantity is $t = \# \text{ of G-orbits of } \mathcal{B}$. Thus $\sum_{g \in G} \text{Fix}(g) = |G|t$ as required.