54. Let \( n > 2 \) and suppose \( d \) is a positive integer which is not the \( n \)-th power of any integer. Let \( \sqrt[n]{d} \) denote the positive real \( n \)-th root of \( d \). Let \( E \) be the Galois closure of \( \mathbb{Q}(\sqrt[n]{d})/\mathbb{Q} \). Show that \( \text{Gal}(E/\mathbb{Q}) \) is non-abelian. Can you generalize this?

55. Let \( \omega = e^{2\pi i / 7} \) and \( L = \mathbb{R} \cap \mathbb{Q}(\omega) \). Show that \( L/\mathbb{Q} \) is Galois with cyclic (hence solvable) Galois group, yet \( L/\mathbb{Q} \) is not a radical extension.

56. Let \( G \) be a compact Hausdorff topological group and let \( L \) be a field with the following properties:

   (i) \( G \subseteq \text{Aut}(L) \) as abstract groups;

   (ii) \( G_x \) (= the stabilizer of \( x \in L \)) is open for every \( x \in L \), i.e. \( L \) is a discrete \( G \)-module.

Prove that \( L/L^G \) is a Galois extension, and the inclusion homomorphism \( i : G \hookrightarrow \text{Aut}(L) \) induces a topological isomorphism \( G \cong \text{Gal}(L/L^G) \). In particular, \( G \) is a profinite group.

57. Show that every profinite group is a Galois group. Hint: First show that any Cartesian product of finite groups is a Galois group. Then use the fact that any profinite group is a closed subgroup of a Cartesian product of finite groups.