58. Let $M$ and $N$ be matrix representatives for $\phi$ and $\psi$, respectively. The Kronecker product of $M$ and $N$ is a matrix representative of $\phi \otimes \psi$. Let $a_1, \ldots, a_n$ be the diagonal entries of $M$ and $b_1, \ldots, b_m$ be the diagonal entries of $N$. Then
\[
\text{tr}(\phi \otimes \psi) = \sum_{i=1}^{n} (a_i \left( \sum_{j=1}^{m} b_j \right)) = \sum_{i,j} a_i b_j = \text{tr}(\phi) \text{tr}(\psi).
\]

59. (Dummit-Foote, 18.1, #9) Suppose that $V \subset H$ is a non-zero $RH_{Q_8}$-stable subspace. Let $a + bi + cj + dk \in V$, and suppose that this element is non-zero. Then
\[
1 = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2} \cdot (a + bi + cj + dk) \in V,
\]
but 1 generates $H$ as an $RH_{Q_8}$-module. Therefore $V = H$ and this representation is irreducible.

61. (Dummit-Foote, 18.3, #6) Write $\psi = \sum n_i \psi_i$ where the $\psi_i$ are irreducible characters of $G$ and $n_i \in \mathbb{Z}_{>0}$. Then $(\psi, \chi_1) = \sum n_i (\psi_i, \chi_1)$ by linearity, but $(\psi_i, \chi_1) = 0$ unless $\psi_i = \chi_1$, in which case $(\psi, \chi_1) = 1$. Therefore $(\psi, \chi_1)$ is equal to the number of times the trivial representation occurs in $\psi$. But this is exactly dim$W$.

(Dummit-Foote, 18.3, #7) (a) The subspaces spanned by the $B_i$ are exactly the $G$-stable subspaces of $V$, so the result follows from Maschke's Theorem.

(b) Let $v_i = \sum a_k e_k \in V_i$ be fixed by all $g \in G$ (the sum runs over all of the $e_k \in B_i$). Because $G$ acts transitively on $B_i$, we can find $g_{k\ell} \in G$ such that $g_{k\ell} \cdot e_k = e_\ell$. Then $v_i = g_{k\ell} \cdot v_i = \cdots + a_k e_\ell + \cdots + a_\ell e_k + \cdots$, so $a_k = a_\ell$ for all $k, \ell$. Therefore $v_i$ is a multiple of $\sum e_k$. 

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(c) We can decompose $W$ as $W = \bigoplus (W \cap V_i)$. The subspace $W \cap V_i$ of $V_i$ is 1-dimensional by part (b), hence $\dim W = \sum \dim W \cap V_i = t$.

(Dummit-Foote, 18.3, #8) The action of $G$ on $\{1, \ldots, n\}$ extends to a representation of $G$ on $\mathbb{C}^n$ by letting $G$ permute the basis elements. Let $\chi$ be the character of this representation and let $W$ be the fixed subspace. By Exercise 6, we have

$$\dim W = (\chi, \chi_1) = \frac{1}{|G|} \sum_g \chi(g).$$

By Exercise 8, we have $\dim W = \text{number of orbits of } G \text{ on the set } \{1, \ldots, n\}$. It is clear from the description of the representation that $\text{Fix}(g) = \chi(g)$ (write down matrices). Putting all of these facts together gives the desired formula.