Solutions to Homework 1

1. (a) The four Sylow 3-subgroups of $A_4$ are all cyclic of order three, and the elements $(123)$, $(124)$, $(134)$, and $(234)$ each generate distinct Sylow 3-subgroups. There is a single Sylow 2-subgroup, which is generated by $(12)(34)$ and $(13)(24)$.

(b) Use Proposition 10 on page 125 of Dummit-Foote.

2. Suppose that $G = \bigcup_{g \in G} gHg^{-1}$. The number of distinct conjugate subgroups of $H$ is equal to $[G : N_G(H)]$, and since each conjugate subgroup has order equal to that of $H$, we see that

$$|H| \cdot \frac{|G|}{|H|} = |G| = |H| \cdot \frac{|G|}{|N_G(H)|}.$$ 

Canceling $|H|$ from the left and right side of the above equation shows that the number of distinct cosets of $H$ is equal to the number of distinct conjugate subgroups. However, $G$ is equal to the union of all the distinct cosets of $H$, all of which intersect trivially and have size $|H|$. But the conjugate subgroups of $H$ all intersect in $\{1\}$, hence their union must contain fewer elements than the union over all of the cosets of $H$. This is a contraction.

3. If $A$ is an abelian group of order 144 then $A = A(2) \oplus A(3)$, where $A(p)$ is the $p$-primary component of $A$. Since $144 = 2^4 \cdot 3^2$, and there are 5 abelian groups of order 16 and 2 of order 9, there are $5 \cdot 2 = 10$ abelian groups of order 144.

4. Let $M$ be any $S^{-1}R$-module, and consider it as an $R$-module via the natural map $R \rightarrow S^{-1}R$. Then there is an isomorphism $S^{-1}M = S^{-1}R \otimes_R M \rightarrow M$ of $S^{-1}R$-modules, given by $(a/s) \otimes_R m \mapsto (a/s)m$.

Now let $0 \rightarrow M' \rightarrow M$ be an exact sequence of $S^{-1}R$-modules. Considering it as an exact sequence of $R$-modules, we tensor with the flat $R$-module $N$ to obtain the exact sequence

$$0 \rightarrow N \otimes_R M' \rightarrow N \otimes_R M.$$ 

Since localizing with respect to $S$ preserves exactness and commutes with tensors, we obtain the exact sequence

$$0 \rightarrow S^{-1}N \otimes_{S^{-1}R} S^{-1}M' \rightarrow S^{-1}N \otimes_{S^{-1}R} S^{-1}M.$$ 

By the remark at the beginning of the solution, this is equivalent to the sequence

$$0 \rightarrow S^{-1}N \otimes_{S^{-1}R} M' \rightarrow S^{-1}N \otimes_{S^{-1}R} M$$

being exact. Therefore $S^{-1}N$ is flat.

5. (a) See Atiyah-MacDonald, Proposition 1.11.
(b) See Atiyah-MacDonald, Proposition 8.3.

6. By the Chinese Remainder Theorem, we have the following isomorphisms of \( \mathbb{C} \)-algebras:

\[
\begin{align*}
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} & \simeq \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}[x]/(x^2 + 1)) \\
& \simeq \mathbb{C}[x]/(x^2 + 1) \\
& \simeq \mathbb{C}[x]/(x + i) \times \mathbb{C}[x]/(x - i) \\
& \simeq \mathbb{C} \times \mathbb{C}.
\end{align*}
\]

7. Since \( A \) satisfies the polynomial \( t^3 - 1 \), we must have that the minimal polynomial \( m(t) \) divides \( t^3 - 1 \). Therefore \( m(t) = t - 1, t^2 + t + 1, \) or \( t^3 - 1 \). If \( m(t) = t - 1 \), then \( A = I \). If \( m(t) = t^2 + t + 1 \), then there is a degree 1 invariant factor dividing \( m(t) \), but this is impossible because \( m(t) \) is irreducible in \( \mathbb{Q}[t] \). If \( m(t) = t^3 - 1 \), then \( m(t) \) is the only invariant factor, so the rational canonical form of \( A \) (and thus the similarity class of \( A \)) is just the companion matrix to \( m(t) \).

8. The module \( R^n \) is free and thus projective. So \( M = \ker(\phi) \oplus R^n \). Thus \( M \) projects onto \( \ker(\phi) \), showing that \( \ker(\phi) \) is a quotient of a finitely generated module, and thus finitely generated.

9. (a) Let \( \mathfrak{m} \) be a maximal ideal of \( R \). The functor \( R/\mathfrak{m} \otimes_R - \) is right exact, so it preserves surjections. Since \( R/\mathfrak{m} \otimes_R R^k = (R/\mathfrak{m})^k \), we have that the map \( (R/\mathfrak{m})^m \rightarrow (R/\mathfrak{m})^n \) is surjective. Since these are vector spaces over the field \( R/\mathfrak{m} \), we see by linear algebra that \( m \geq n \).

(b) (This is the solution given in class.) Let \( A : R^n \rightarrow R^m \) denote the map in question, and suppose that \( n > m \). We can extend \( A \) to a map \( A' : R^n \rightarrow R^n \oplus R^{n-m} \simeq R^n \) by, for example, adding \( n - m \) rows of zeros to a matrix representation of \( A \). Note that if \( A \) is injective, so is \( A' \). Since \( n - m > 0 \), we have \( \det(A') = 0 \), so by a homework problem last semester, there exists \( v \in R^n, v \neq 0 \), such that \( A' v = 0 \). But then \( Av = 0 \), contradicting the injectivity of \( A \).