1. (10 points) Let $R$ be a commutative ring with unit. Let $V, W$ be $R$-modules. In class we proposed a definition of multiplication on the $R$-module $E(V) \otimes_R E(W)$, which should make it into an $R$-algebra (where $R$ belongs to the center). On generators, the multiplication we proposed was given by defining

$$[(v_1 \wedge \cdots \wedge v_k) \otimes (w_1 \wedge \cdots \wedge w_l)] [(x_1 \wedge \cdots \wedge x_t) \otimes (y_1 \wedge \cdots \wedge y_s)]$$

to be

$$(-1)^{lt} (v_1 \wedge \cdots \wedge v_k \wedge x_1 \wedge \cdots \wedge x_t) \otimes (w_1 \wedge \cdots \wedge w_l \wedge y_1 \wedge \cdots \wedge y_s).$$

Show that this does indeed give a well-defined product on $E(V) \otimes_R E(W)$, making into an $R$-algebra such that $R$ belongs to the center. Warning: in this general situation, $E(V)$ and $E(W)$ are not freely generated by the above generators, so you have to take care in proving the product is well-defined.

2. (10 points) Following the discussion from class, construct the maps $E(V) \otimes_R E(W) \to E(V \oplus W)$ and $E(V \oplus W) \to E(V) \otimes_R E(W)$, and verify they are mutually inverse $R$-algebra homomorphisms.

3. (10 points) Dummit-Foote, 11.5, #1.


5. (a) (5 points) Let $R$ be a commutative ring with identity, and for an $R$-module $M$ define the $R$-module $M^* = \text{Hom}_R(M, R)$. Define a natural $R$-linear homomorphism $M \otimes_R M^* \to R$. Then use adjointness (cf. Dummit-Foote, 10.5 Thm. 43) to construct a natural morphism $M \to M^{**}$.

(b) (5 points) Discuss whether $M \to M^{**}$ is an isomorphism, injective, or surjective. Can you find examples of rings $R$ and certain conditions on $M$ such that some or all of these properties hold, and other examples showing that one or more can fail?
6. Let $\mathcal{A}$ be an abelian category. Recall that this means the following:

- $\mathcal{A}$ is an additive category: for objects $A, B, C$, the set $\text{Hom}(A, B)$ is an abelian group and the composition map $\text{Hom}(A, B) \times \text{Hom}(B, C) \to \text{Hom}(A, C)$ is a group homomorphism;
- in $\mathcal{A}$ we can form direct sums, kernels, and cokers (each of which satisfies a certain universal property, discussed in class). Also, if we define $\text{im}(f : A \to B) := \ker(B \to \text{cok}(f))$ and $\text{coim}(f : A \to B) := \text{cok}(\ker(f) \to A)$, then the canonical map

$$\text{coim}(f) \to \text{im}(f)$$

is an isomorphism.

(a) (5 points) If $\mathcal{A}$ is an abelian category, define the category $\mathbf{C}(\mathcal{A})$ of complexes over $\mathcal{A}$.

(b) (15 points) Show that $\mathbf{C}(\mathcal{A})$ is an abelian category.