Solutions to Homework 2

1. Consider the map \( V^k \times W^\ell \times V^t \times W^s \to V^{k+t} \times W^{\ell+s} \) given by
\[
(v_1, \ldots, v_k, w_1, \ldots, w_\ell, x_1, \ldots, x_t, y_1, \ldots, y_s) \mapsto (-1)^{\ell t}(v_1, \ldots, v_k, x_1, \ldots, x_t, w_1, \ldots, w_\ell, y_1, \ldots, y_s).
\]
On composing with the canonical map \( V^{k+t} \times W^{\ell+s} \to \bigwedge^{k+t} V \otimes_R \bigwedge^{\ell+s} W \) we get a map
\[
V^k \times W^\ell \times V^t \times W^s \to \bigwedge^{k+t} V \otimes_R \bigwedge^{\ell+s} W
\]
given by
\[
(v_1, \ldots, v_k, w_1, \ldots, w_\ell, x_1, \ldots, x_t, y_1, \ldots, y_s) \mapsto (-1)^{\ell t}(v_1 \wedge \ldots \wedge v_k \wedge x_1 \wedge \ldots \wedge x_t) \otimes (w_1 \wedge \ldots \wedge w_\ell \wedge y_1 \wedge \ldots \wedge y_s).
\]
This map clearly alternating and multilinear with respect to the \( v_i, w_i, x_i, \) and \( y_i \). Therefore it induces a map
\[
\bigwedge^{k+t} V \otimes_R \bigwedge^{\ell+s} W \to \bigwedge^{k+t} V \otimes R \bigwedge^{\ell+s} W
\]
which, by multilinearity, induces the required map
\[
\left( \bigwedge^k V \otimes_R \bigwedge^\ell W \right) \times \left( \bigwedge^t V \otimes_R \bigwedge^s W \right) \to \bigwedge^{k+t} V \otimes R \bigwedge^{\ell+s} W.
\]

2. Consider the map \( V \times W \to \bigwedge(V \oplus W) \) given by \((v, w) \mapsto (v, 0) \wedge (0, w)\). It induces an \( R \)-algebra homomorphism
\[
f : \bigwedge(V) \otimes_R \bigwedge(W) \to \bigwedge(V \oplus W)
\]
such that \( f(v \otimes w) = (v, 0) \wedge (0, w) \).

To go the other way, consider the map \( V \oplus W \to \bigwedge(V) \otimes_R \bigwedge(W) \) defined by \((v, w) \mapsto v \otimes 1 + 1 \otimes w\). It is bilinear and alternating, and thus induces an \( R \)-algebra homomorphism
\[
g : \bigwedge(V \oplus W) \to \bigwedge(V) \otimes_R \bigwedge(W)
\]
such that \( g(v, w) = v \otimes 1 + 1 \otimes w \). We wish to show that \( f \) and \( g \) are mutually inverse; because \( f \) and \( g \) are algebra homomorphisms, it suffices to check this on algebra generators.

As an \( R \)-algebra, \( \bigwedge(V) \otimes_R \bigwedge(W) \) is generated by \( 1 \in R \) and elements of the form \( v \otimes 1 + 1 \otimes w \). The \( R \)-algebra \( \bigwedge(V \oplus W) \) is generated by \( 1 \in R \) and elements \((v, w) \in V \oplus W\). Clearly \( f(g(1)) = 1 \) and \( g(f(1)) = 1 \). We also have
\[
g(f(v \otimes 1 + 1 \otimes w)) = g((v, 0) + (0, w)) = g(v, 0) + g(0, w) = v \otimes 1 + 1 \otimes w
\]
and

\[ f(g(v, w)) = f(v \otimes 1 + 1 \otimes w) = (v, 0) + (0, w) = (v, w) \]

so we are done.

3. Let \( M \) be generated as an \( R \)-module by \( x \in M \). We must show that \( C(M) \), which is the submodule of \( T(M) \) generated by elements of the form \( m_2 \otimes m_1 - m_1 \otimes m_2 \), is zero. Pick arbitrary \( m_1, m_2 \in M \). Because \( M \) is cyclic, we can write \( m_1 = r_1 x \) and \( m_2 = r_2 x \) for some \( r_1, r_2 \in R \). Then

\[ m_2 \otimes m_1 - m_1 \otimes m_2 = r_2 x \otimes r_1 x - r_1 x \otimes r_2 x = (r_1 r_2 x) \otimes x - (r_1 r_2 x) \otimes x = 0. \]

Thus \( C(M) = 0 \) and \( T(M) = S(M) \).

4. We can identify \( S^2(V) \) with the symmetric tensors of \( T^2(V) \), and the symmetric tensors of \( T^2(V) \) are generated by elements of the form \( v \otimes w + w \otimes v \). Similarly, we can identify \( \Lambda^2(V) \) with the alternating tensors of \( T^2(V) \), which are generated by elements of the form \( v \otimes w - w \otimes v \) (see the discussion on pages 451-453 of the text; we are using that \( 2 \neq 0 \) in \( F \)).

Define a map \( V \times V \to S^2(V) \oplus \Lambda^2(V) \) by \( (v, w) \mapsto (v \otimes w + w \otimes v, v \otimes w - w \otimes v) \). It is easy to check that this is bilinear, hence induces a map

\[ \phi : V \otimes_F V \to S^2(V) \oplus \Lambda^2(V) \]

sending \( v \otimes w \mapsto (v \otimes w + w \otimes v, v \otimes w - w \otimes v) \). We must prove that \( \phi \) is injective and surjective.

Suppose that \( \phi(v \otimes w) = 0 \). Then \( v \otimes w = -w \otimes v \) and \( v \otimes w = w \otimes v \). Thus \( v \otimes w = -v \otimes w \) and since \( 2 \neq 0 \) in \( F \), this implies that \( v \otimes w = 0 \). Thus \( \phi \) is injective.

To see that \( \phi \) is surjective, note that \( S^2(V) \oplus \Lambda^2(V) \) is generated by elements of the form \( (v \otimes w + w \otimes v, 0) \) and \( (0, v \otimes w - w \otimes v) \), so we only need to prove that these are in the image of \( \phi \). It is easy to check that \( \phi(\frac{1}{2}(v \otimes w + w \otimes v)) = (v \otimes w + w \otimes v, 0) \) and \( \phi(\frac{1}{2}(v \otimes w - w \otimes v)) = (0, v \otimes w - w \otimes v) \). Thus \( \phi \) is surjective and hence an isomorphism.

5. (extra credit) (a) The category \( C(A) \) is defined to have objects of the form

\[ \ldots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \to \ldots \]

where \( A^i \in A \) for all \( i \), and \( d^i \) is a morphism in \( A \) such that \( d^i \circ d^{i-1} = 0 \) for all \( i \). We use the notation \( (A^*, d_A) \) or just \( A^* \) to denote the above complex.

A morphism \( f : A^* \to B^* \) is a collection of morphisms \( f_i : A^i \to B^i \), such that \( f_{i+1} \circ d^A_A = d^B_B \circ f_i \) for all \( i \). Composition of morphisms is done at each \( i \); that is, \( (f \circ g)_i = f_i \circ g_i \). It is easy to check
that $f \circ g$ is a morphism in $\mathbf{C}(\mathcal{A})$.

(b) (sketch) The formation of direct sums, kernels, and cokernels is done at each $i$. For example, if $f : A^\bullet \to B^\bullet$, then $(\ker f)^i = \ker(f^i)$. It is not hard to see that this forms a complex, and to prove its universal property we apply the universal property of $\ker(f^i)$ at each $i$.

To see that the natural map $\text{coim}(f) \to \text{im}(f)$ is an isomorphism, note that because $\mathcal{A}$ is abelian, we have isomorphisms $\phi^i : \text{coim}(f^i) \tilde{\to} \text{im}(f^i)$. Once one knows that the $\phi^i$ fit together to form a chain map, we have our isomorphism.