Solutions to Homework 5

18. The \( \mathbb{Z} \)-module \( \mathbb{Z}/m\mathbb{Z} \) has projective resolution \( 0 \rightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow 0 \). Tensoring with \( \mathbb{Z}/n\mathbb{Z} \) yields the exact sequence

\[
\left( \mathbb{Z}/n\mathbb{Z} \right) \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{1 \otimes m} \left( \mathbb{Z}/n\mathbb{Z} \right) \otimes_{\mathbb{Z}} \mathbb{Z} \rightarrow \left( \mathbb{Z}/n\mathbb{Z} \right) \otimes_{\mathbb{Z}} \left( \mathbb{Z}/m\mathbb{Z} \right) \rightarrow 0
\]

and \( \text{Tor}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \) is the kernel of the first map. Under the natural isomorphism \( (\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z} \simeq \mathbb{Z}/n\mathbb{Z} \), the map \( 1 \otimes m \) becomes multiplication by \( m \) in \( \mathbb{Z}/n\mathbb{Z} \). The kernel of this map is the set of elements of order dividing \( m \), which is \( \mathbb{Z}/(m,n)\mathbb{Z} \subseteq \mathbb{Z}/n\mathbb{Z} \).

19. One sees easily that \( \text{Ext}^i_{\mathbb{R}}(R/I, M) = 0 \) for \( i > 1 \). So we only need to compute \( \text{Ext}^1_{\mathbb{R}}(R/I, M) \).

Applying \( \text{Hom}_{\mathbb{R}}(\cdot, M) \) to the exact sequence, we get that \( \text{Ext}^1_{\mathbb{R}}(R/I, M) \) is the cokernel of the (surjective) map \( \text{Hom}_{\mathbb{R}}(R, M) \rightarrow \text{Hom}_{\mathbb{R}}(R, M) \) induced by multiplication by \( a \) on \( R \). Under the natural isomorphism \( \text{Hom}_{\mathbb{R}}(R, M) \simeq M \), this map becomes \( m \mapsto \rightarrow am \). Thus \( \text{Ext}^1_{\mathbb{R}}(R/I, M) \simeq M/aM \). Note that \( \text{Ext}^0_{\mathbb{R}}(R/I, M) = \text{Hom}_{\mathbb{R}}(R/I, M) \) is the set of \( m \in M \) that are killed by \( a \).

20. (Dummit-Foote 17.1, #22) Let \( P \rightarrow A \) be a projective resolution of \( A \) as an \( \mathbb{R} \)-module. The complex \( P \otimes_{\mathbb{R}} S \) is exact because \( S \) is a flat \( \mathbb{R} \)-module. Since tensoring commutes with direct sums, it preserves projectives. Thus \( P \otimes_{\mathbb{R}} S \) is a projective resolution of \( A \otimes_{\mathbb{R}} S \). Furthermore, note that we have isomorphisms \( (P \otimes_{\mathbb{R}} S) \otimes_{S} B \simeq P \otimes_{\mathbb{R}} (S \otimes_{S} B) \simeq P \otimes_{\mathbb{R}} B \) of complexes. It follows from these observations that

\[
\text{Tor}^S_n(A \otimes_{\mathbb{R}} S, B) = H_n((P \otimes_{\mathbb{R}} S) \otimes_{S} B)
\]

\[
\simeq H_n(P \otimes_{\mathbb{R}} B)
\]

\[
= \text{Tor}^R_n(A, B).
\]

(Dummit-Foote 17.1, #23) As \( D^{-1}R \) is flat as an \( \mathbb{R} \)-module, we have that

\[
\text{Tor}^R_n(A, D^{-1}B) \simeq \text{Tor}^R_n(D^{-1}A, D^{-1}B)
\]

by the previous problem (with \( S = D^{-1}R \) and \( D^{-1}B \) in place of \( B \)). Let \( P \rightarrow B \) be a projective resolution of \( B \) as an \( \mathbb{R} \)-module. Because \( D^{-1}R \) is a flat \( \mathbb{R} \)-module, we have isomorphisms

\[
D^{-1}\text{Tor}^R_n(A, B) \simeq H_n(D^{-1}A \otimes_{\mathbb{R}} P)
\]

\[
= \text{Tor}^R_n(D^{-1}A, B).
\]

As interchanging the roles of \( A \) and \( B \) does not change \( D^{-1}\text{Tor}^R_n(A, B) \), we have that

\[
\text{Tor}^R_n(D^{-1}A, B) \simeq \text{Tor}^R_n(D^{-1}B, A)
\]

\[
\simeq \text{Tor}^R_n(A, D^{-1}B)
\]
and so \( D^{-1}\text{Tor}^R_n(A, B) \simeq \text{Tor}^R_n(A, D^{-1}B) \simeq \text{Tor}^{D^{-1}R}_n(D^{-1}A, D^{-1}B) \) as desired.

(Dummit-Foote 17.1, #24) Let \( M \) and \( N \) be \( R \)-modules. By the previous problem, we have that \( \text{Tor}^R_n(M, N) = 0 \iff \text{Tor}^R_n(M, N)_m = 0 \) for all maximal ideals \( m \) of \( R \iff \text{Tor}^R_n(M_m, N_m) = 0 \) for all \( m \). By the characterization of flatness in terms of Tor, it follows that \( M \) is flat \( \iff \) \( M_m \) is flat for all \( m \).

21. (a) The implications \((i) \Rightarrow (ii) \Rightarrow (iii)\) are proven in section 10.5 of Dummit-Foote, and the implication \((iii) \Rightarrow (iv)\) is Proposition 16 of section 17.1 of Dummit-Foote.

(b) Following the hint, let \( N \) be the submodule of \( M \) generated by \( x_1, \ldots, x_n \). Then \( N + mM = M \), so by Nakayama’s Lemma, we have \( M = N \). We have an exact sequence \( 0 \rightarrow K \rightarrow R^n \rightarrow M \rightarrow 0 \) where \( e_i \mapsto x_i \) and \( K \) is defined to be the kernel. As \( \text{Tor}^R_1(M, k) = 0 \), this sequence stays exact when tensored with \( k \). Upon tensoring with \( k \), the second map becomes a surjective map of finite-dimensional vector spaces over \( k \), which is therefore bijective by linear algebra. Hence \( K \otimes_R k = 0 \). But \( K \otimes_R k \simeq K/mK \), hence \( K = mK \) and it follows from Nakayama’s Lemma that \( K = 0 \). Therefore \( R^n \simeq M \) and \( M \) is free.

(c) (i) The abelian group \( \mathbb{Z} \) is a \( \mathbb{Z} \oplus \mathbb{Z} \)-module via the action \( (x_1, x_2) \cdot m = x_1m \). It is clearly finitely generated, and it is projective because it is a direct summand of the free module \( \mathbb{Z} \oplus \mathbb{Z} \). However, it cannot be free because its rank as an abelian group is too small.

(ii) The \( \mathbb{Z} \)-module \( \mathbb{Q} \) is flat by the argument on Dummit-Foote page 401. However, it is not projective by Exercise 8 of section 10.5 of Dummit-Foote.

(iii) Let \( R = \mathbb{Z} \), and let \( M = \mathbb{Z}/p\mathbb{Z} \) have its natural \( \mathbb{Z} \)-module structure, where \( p \) is a prime. Pick a prime \( \ell \neq p \) and let \( m = \ell\mathbb{Z} \). Then \( \text{Tor}^R_1(M, R/m) = \text{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/\ell\mathbb{Z}) = 0 \) by Exercise 18. However, \( \mathbb{Z}/p\mathbb{Z} \) is not flat, because tensoring with it does not preserve the exactness of the sequence \( 0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0 \).