Solutions to Homework 7

27. (Dummit-Foote 13.2 #21) Let $\alpha = a + b\sqrt{D}$. Considering K as a $\mathbf{Q}$-vector space with basis $1, \sqrt{D}$, we see that $\alpha \cdot 1 = a + b\sqrt{D}$ and $\alpha \cdot \sqrt{D} = bD + a\sqrt{D}$. Therefore the matrix for multiplication by α is

$$\begin{pmatrix} a & bD \\ b & a \end{pmatrix}.$$

It is straightforward to verify that $\alpha \mapsto$ (the above matrix) is a ring homomorphism, and it is clearly non-zero. Since K is a field it has no non-zero ideals and thus our map is injective. Since it is obviously surjective, we are done.

(Dummit-Foote 13.2 #22) Let $\{\alpha_i\}$ be a basis for K_1 over F, and let $\{\beta_j\}$ be a basis for K_2 over F. Then $\{\alpha_i \otimes \beta_j\}$ is a basis for $K_1 \otimes_F K_2$ over F. Define a map $\phi : K_1 \otimes_F K_2 \to K_1K_2$ by $\phi(\alpha_i \otimes \beta_j) = \alpha_i\beta_j$, and extend it by linearity. It is easy to check that ϕ is an F-algebra homomorphism. The map ϕ is surjective because the elements $\alpha_i\beta_j$ span K_1K_2 as an F-vector space.

($\Leftarrow$) By assumption the F-vector spaces $K_1 \otimes_F K_2$ and K_1K_2 have the same dimension over F, namely $[K_1 : F][K_2 : F]$. Thus by linear algebra ϕ is injective and hence an isomorphism. Therefore $K_1 \otimes_F K_2$ is isomorphic to the field K_1K_2.

($\Rightarrow$) Conversely, if $K_1 \otimes_F K_2$ is a field then it has no non-zero ideals so $\ker(\phi) = 0$. Therefore ϕ is an isomorphism and $[K_1K_2 : F] = [K_1 \otimes_F K_2 : F] = [K_1 : F][K_2 : F]$.

28. (Dummit-Foote 13.4 #5) ($\Rightarrow$) Suppose K is a splitting field. Let $f \in F[X]$ be irreducible, and let α be a root of f in K. Let β be an arbitrary root of f. By Theorems 8 and 27, there is a commutative diagram

$$\begin{tikzcd}
K(\alpha) \arrow[r, "\sim"] & K(\beta) \\
F(\alpha) \arrow[u] \arrow[r, "\sim"] & F(\beta) \arrow[u] \\
F \arrow[u] \arrow[r, "\mathrm{id}"] & F \arrow[u]
\end{tikzcd}$$

of maps of fields. But $\alpha \in K$ implies that $[K(\beta) : K] = [K(\alpha) : K] = 1$ and thus $\beta \in K$ as desired.

($\Leftarrow$) Because K/F is a finite extension, we have $K = F(\alpha_1, \ldots, \alpha_n)$ for some $\alpha_i \in K$. Let f_i be the minimal polynomial of α_i, then f_i splits completely in K by assumption. Therefore K is the splitting field of $\prod_i f_i$.

(Dummit-Foote 13.4 #6) (a) Let K_1 be the splitting field of f_1 and let K_2 be the splitting field of f_2. Let K be the splitting field of $f = f_1f_2$. We will show that $K = K_1K_2$. It is clear that $K \subseteq K_1K_2$ because all of the roots of f lie in K_1K_2. Conversely, if K splits f then it certainly splits f_1, hence $K_1 \subseteq K$. Similarly, $K_2 \subseteq K$. Therefore $K = K_1K_2$.

(b) We will use the previous exercise. Let f be a polynomial with a root α in $K_1 \cap K_2$. Then $\alpha \in K_1$ and $\alpha \in K_2$ and because each of these is a splitting field, all of the other roots of f must be in both K_1 and K_2. Therefore f splits completely in $K_1 \cap K_2$, which proves that $K_1 \cap K_2$ is a splitting field.

30. Let $\zeta = \sqrt{2}/2 + i\sqrt{2}/2$, then ζ is a primitive eighth root of unity. If $\sqrt[8]{2}$ denotes the real positive eighth root of 2, then $K = \mathbf{Q}(\sqrt[8]{2}, \zeta) = \mathbf{Q}(\sqrt[8]{2}, i)$. As $\mathbf{Q}(\sqrt[8]{2}) \subseteq \mathbf{R}$, we have $i \notin \mathbf{Q}(\sqrt[8]{2})$ so $\mathbf{Q} \subsetneq \mathbf{Q}(\sqrt[8]{2}) \subsetneq K$. Since $[\mathbf{Q}(\sqrt[8]{2}) : \mathbf{Q}] = 8$ and $[K : \mathbf{Q}(\sqrt[8]{2})] = 2$, we have $[K : \mathbf{Q}] = 8 \cdot 2 = 16$.

31. Let $f \in K[X_1, \ldots, X_n]$ and suppose f is in the intersection of all of the maximal ideals of $K[X_1, \ldots, X_n]$. Then $1 + fg$ is a unit (hence in $K^\times$) for all $g \in K[X_1, \ldots, X_n]$. Setting $g = 1$ we have that f must be constant, and as it is contained in every maximal ideal, we must therefore have $f = 0$.

32. We have a tower of fields $K \subseteq K(a^2) \subseteq K(a)$. The degree $[K(a) : K]$ is equal to the degree of the minimal polynomial of a, which is odd. By multiplicativity of degrees of field extensions, we see that $[K(a) : K(a^2)]$ must also be odd. However, a satisfies the polynomial $X^2 - a^2 \in K(a^2)[X]$ which is degree two. Therefore $[K(a) : K(a^2)] \leq 2$, so we have $[K(a) : K(a^2)] = 1$ and thus $K(a) = K(a^2)$.