27. (Dummit-Foote 13.2 #21) Let $\alpha = a + b\sqrt{D}$. Considering $K$ as a $\mathbb{Q}$-vector space with basis $1, \sqrt{D}$, we see that $\alpha \cdot 1 = a + b\sqrt{D}$ and $\alpha \cdot \sqrt{D} = bD + a\sqrt{D}$. Therefore the matrix for multiplication by $\alpha$ is

$$
\begin{pmatrix}
 a & bD \\
 b & a
\end{pmatrix}.
$$

It is straightforward to verify that $\alpha \mapsto (\text{the above matrix})$ is a ring homomorphism, and it is clearly non-zero. Since $K$ is a field it has no non-zero ideals and thus our map is injective. Since it is obviously surjective, we are done.

(Dummit-Foote 13.2 #22) Let $\{\alpha_i\}$ be a basis for $K_1$ over $F$, and let $\{\beta_j\}$ be a basis for $K_2$ over $F$. Then $\{\alpha_i \otimes \beta_j\}$ is a basis for $K_1 \otimes_F K_2$ over $F$. Define a map $\phi : K_1 \otimes_F K_2 \to K_1 K_2$ by $\phi(\alpha_i \otimes \beta_j) = \alpha_i \beta_j$, and extend it by linearity. It is easy to check that $\phi$ is an $F$-algebra homomorphism. The map $\phi$ is surjective because the elements $\alpha_i \beta_j$ span $K_1 K_2$ as an $F$-vector space.

$(\Leftrightarrow)$ By assumption the $F$-vector spaces $K_1 \otimes_F K_2$ and $K_1 K_2$ have the same dimension over $F$, namely $[K_1 : F][K_2 : F]$. Thus by linear algebra $\phi$ is injective and hence an isomorphism. Therefore $K_1 \otimes_F K_2$ is isomorphic to the field $K_1 K_2$.

$(\Rightarrow)$ Conversely, if $K_1 \otimes_F K_2$ is a field then it has no non-zero ideals so $\ker(\phi) = 0$. Therefore $\phi$ is an isomorphism and $[K_1 K_2 : F] = [K_1 \otimes_F K_2 : F] = [K_1 : F][K_2 : F]$.

28. (Dummit-Foote 13.4 #5) $(\Rightarrow)$ Suppose $K$ is a splitting field. Let $f \in F[X]$ be irreducible, and let $\alpha$ be a root of $f$ in $K$. Let $\beta$ be an arbitrary root of $f$. By Theorems 8 and 27, there is a commutative diagram

$$
\begin{array}{ccc}
K(\alpha) & \sim & K(\beta) \\
\uparrow & & \uparrow \\
F(\alpha) & \sim & F(\beta) \\
\uparrow & & \uparrow \\
F & \xrightarrow{\text{id}} & F
\end{array}
$$

of maps of fields. But $\alpha \in K$ implies that $[K(\beta) : K] = [K(\alpha) : K] = 1$ and thus $\beta \in K$ as desired.

$(\Leftrightarrow)$ Because $K/F$ is a finite extension, we have $K = F(\alpha_1, \ldots, \alpha_n)$ for some $\alpha_i \in K$. Let $f_i$ be the minimal polynomial of $\alpha_i$, then $f_i$ splits completely in $K$ by assumption. Therefore $K$ is the splitting field of $\prod_i f_i$.

(Dummit-Foote 13.4 #6) (a) Let $K_1$ be the splitting field of $f_1$ and let $K_2$ be the splitting field of $f_2$. Let $K$ be the splitting field of $f = f_1 f_2$. We will show that $K = K_1 K_2$. It is clear that $K \subseteq K_1 K_2$ because all of the roots of $f$ lie in $K_1 K_2$. Conversely, if $K$ splits $f$ then it certainly splits $f_1$, hence $K_1 \subseteq K$. Similarly, $K_2 \subseteq K$. Therefore $K = K_1 K_2$. 

1
(b) We will use the previous exercise. Let $f$ be a polynomial with a root $\alpha$ in $K_1 \cap K_2$. Then $\alpha \in K_1$ and $\alpha \in K_2$ and because each of these is a splitting field, all of the other roots of $f$ must be in both $K_1$ and $K_2$. Therefore $f$ splits completely in $K_1 \cap K_2$, which proves that $K_1 \cap K_2$ is a splitting field.

30. Let $\zeta = \sqrt{2}/2 + i \sqrt{2}/2$, then $\zeta$ is a primitive eighth root of unity. If $\sqrt[8]{2}$ denotes the real positive eighth root of 2, then $K = \mathbb{Q}(\sqrt[8]{2}, \zeta) = \mathbb{Q}(\sqrt[8]{2}, i)$. As $\mathbb{Q}(\sqrt[8]{2}) \subseteq \mathbb{R}$, we have $i \not\in \mathbb{Q}(\sqrt[8]{2})$ so $\mathbb{Q} \subsetneq \mathbb{Q}(\sqrt[8]{2}) \subsetneq K$. Since $[\mathbb{Q}(\sqrt[8]{2}) : \mathbb{Q}] = 8$ and $[K : \mathbb{Q}(\sqrt[8]{2})] = 2$, we have $[K : \mathbb{Q}] = 8 \cdot 2 = 16$.

31. Let $f \in K[X_1, \ldots, X_n]$ and suppose $f$ is in the intersection of all of the maximal ideals of $K[X_1, \ldots, X_n]$. Then $1 + fg$ is a unit (hence in $K^\times$) for all $g \in K[X_1, \ldots, X_n]$. Setting $g = 1$ we have that $f$ must be constant, and as it is contained in every maximal ideal, we must therefore have $f = 0$.

32. We have a tower of fields $K \subseteq K(a^2) \subseteq K(a)$. The degree $[K(a) : K]$ is equal to the degree of the minimal polynomial of $a$, which is odd. By multiplicativity of degrees of field extensions, we see that $[K(a) : K(a^2)]$ must also be odd. However, $a$ satisfies the polynomial $X^2 - a^2 \in K(a^2)[X]$ which is degree two. Therefore $[K(a) : K(a^2)] \leq 2$, so we have $[K(a) : K(a^2)] = 1$ and thus $K(a) = K(a^2)$. 