Calculus 130, section 2.1-2.2 Exponential and Logarithmic Functions

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Take a piece of paper, and fold it in half. You’ve doubled the number of layers—from 1 to 2. Fold it in half again, and you’ve once again doubled the layers—from 2 to 4. After the next fold you’d have 8 layers. After the next, 16. Then 64, 128, 256, 512, 1024, etc. After just 10 foldings, we have the paperback edition of just the first book of Lord of the Rings that we’re attempting to fold in half.

When \( x \) is the base, we have a power function. When \( x \) is the exponent we have an exponential function. The scenario above illustrates the exponential function \( y = 2^x \).

If we compare the graph of \( y = x^2 \) to the graph of \( y = 2^x \), we can see that for positive values of \( x \) the exponential function grows much more quickly than the power function. (Thus our difficulty in folding a piece of paper in half successive times.) Eventually, we’ll be interested in finding a way to describe the slope of an exponential graph, i.e. a way to find its derivative.

Exponential functions have many applications because they model many kinds of growth and shrinking: e.g. populations, bank deposits, radioactive decay.

Examples A: Sketch the graphs of the following functions, using translations and shifts.

\[
\begin{align*}
y &= 2^{x-1} \\
y &= 2^x - 1 \\
y &= -2^x \\
y &= \left(\frac{1}{2}\right)^x
\end{align*}
\]

Functions with the basic form \( y = b^x \) are actually a family of functions. We’ll consider only values for \( b \) that are positive. (Negative values of \( b \) are extremely problematic, since even and odd values of \( x \) would cause \( y \) to fluctuate between positive and negative.)

Consider the functions \( y = 10^x, y = 5^x, y = 3^x, y = 2^x, y = 1.1^x \) pictured in the graph to the right. Note first the similarities: \( b^0 = 1 \) for all values of \( b \neq 0 \), so \( (0, 1) \) makes a good reference point. Each of the basic exponential functions has a horizontal asymptote \( y = 0 \). The graphs also have similar shape—the major difference is slope of the curve at specific values of \( x \). Note that at \( x = 0 \) slope of \( y = 10^x \) is steepest; slope of \( y = 1.1^x \) is most shallow.
All of the usual properties of exponents apply to exponential functions:

\[
\begin{align*}
\frac{b^x}{b^y} &= b^{x-y} & b^x b^y &= b^{x+y} & \frac{1}{b^y} &= b^{-y} & (b^x)^y &= b^{xy} & a^x b^x &= (ab)^x & \frac{a^x}{b^x} &= \left(\frac{a}{b}\right)^x.
\end{align*}
\]

Examples B: We can use these properties to simplify expressions and solve equations.

Simplify \(2^{x-1} \cdot 8^{x+3}\). \textit{Answer:} \(2^{4x+8} = 2^8 \cdot 2^{4x}\)

Simplify \(\frac{2^{x-1}}{8^{x+3}}\). \textit{Answer:} \(2^{-2x-10} = \frac{1}{2^{10}} \cdot 2^{-2x}\)

Solve \(2^{x^2+3} = 16\). \textit{Answer:} \(x = \pm 1\)

Solve \((\sqrt{3})^x = \frac{1}{27}\). \textit{Answer:} \(x = -6\)

Solve \(\frac{1}{2} = 1 - 2^{x+1}\). \textit{Answer:} \(x = -2\)
Example C: You deposit $100 into a certificate of deposit which pays 5% each year on the balance current at the time. Find an equation to describe the growth of your money.

Define $A(t) =$ amount of money accumulated after $t$ years. The table below summarizes the growth over 5 years.

<table>
<thead>
<tr>
<th>$T$</th>
<th>interest earned</th>
<th>$A =$ accumulated amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$A = 100(1 + 0.05)^0$</td>
</tr>
<tr>
<td>1</td>
<td>0.05 * [100]</td>
<td>$A = 100(1 + 0.05)$</td>
</tr>
<tr>
<td>2</td>
<td>0.05 * [100(1 + 0.05)]</td>
<td>$A = 100(1 + 0.05)^2$</td>
</tr>
<tr>
<td>3</td>
<td>0.05 * [100(1 + 0.05)^2]</td>
<td>$A = 100(1 + 0.05)^3$</td>
</tr>
<tr>
<td>4</td>
<td>0.05 * [100(1 + 0.05)^3]</td>
<td>$A = 100(1 + 0.05)^4$</td>
</tr>
<tr>
<td>5</td>
<td>0.05 * [100(1 + 0.05)^4]</td>
<td>$A = 100(1 + 0.05)^5$</td>
</tr>
</tbody>
</table>

We can use the pattern to state a general formula for interest added annually for $n$ years:

$$A(10) = 100(1 + 0.05)^{10} \approx 162.89.$$ 

If the interest was compounded quarterly, the 5% annual rate would be divided up among the four quarters, and the number of interest calculations would be $n = 4(10)$: $A(10) = 100\left(1 + \frac{0.05}{4}\right)^{40} \approx 164.36$.

For interest compounded monthly, we’d have: $A(10) = 100\left(1 + \frac{0.05}{12}\right)^{120} \approx 164.70$.

For interest compounded daily, we’d have: $A(10) = 100\left(1 + \frac{0.05}{365}\right)^{3650} \approx 164.87$.

For different principals, $P$, rates of interest, $r$, compounding periods, $m$, and numbers of years, $t$, we can generalize: $A(P, r, m, t) = P\left(1 + \frac{r}{m}\right)^{mt}$.

Using this formula we could recalculate our balance compounding every hour, second, or fraction of a second. What happens if we increase the number of times interest is calculated and approach infinity?

An “alternate” way of defining $e$ is $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

If we now take our generalized formula, and replace $\frac{r}{m} = \frac{1}{n}$ we have:

$$A(P, r, t) = \lim_{n \to \infty} P\left(1 + \frac{1}{n}\right)^{nrt} = P\left[\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n\right]^{rt} = Pe^{rt}.$$
This “continuous compounding” formula for money also models some types of biological growth. We’ll investigate applications in section 2.3.

Example C again: For an initial balance of $100 and an annual interest rate of 5% compounded continuously over 10 years, calculate the closing balance rounded to the nearest penny. Answer: $164.87

If you are calculating interest on a hand calculator, the continuous compounding formula is much easier to use. If you are working at a bank or investment firm, doing a massive number of this type of calculation, the continuous compounding formula uses a lot less computer time and memory.

The number $e$ is Euler’s number. Like $\pi$ or $\sqrt{2}$, $e$ is an irrational number. The value of $e$ is approximately 2.717. The corresponding function, $y = e^x$, is called the natural exponential function.

As a side note, this is only one way to define $e$ and approximate its value. There are other definitions, including

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

(Here begins section 2.2)

Given a function $f$ and its inverse, $f^{-1}$, the following will always be true:
1. If $f(a) = b$, then $f^{-1}(b) = a$ (This fact and the statement in point #2 below is actually the same information.)
2. If $(a, b)$ is a point on the graph of $f$, then $(b, a)$ will be on the graph of $f^{-1}$.
3. The domain of $f$ is the range of $f^{-1}$, and the range of $f$ is the domain of $f^{-1}$.
4. $f \circ f^{-1} = x$ and $f^{-1} \circ f = x$. To show that two functions are inverses you must do both compositions.
5. The graph of $f$ and the graph of $f^{-1}$ are symmetric with respect to the line $y = x$.

A logarithm function is a constructed inverse for an exponential function. The natural logarithm function, $y = \ln(x)$, is the inverse of the natural exponential function, $f(x) = e^x$. Applying the above:

1. For one example, $f(0) = e^0 = 1$ and $f^{-1}(1) = \ln(1) = 0$.
2. For one example, $(0, 1)$ is on the graph of $f(x) = e^x$ and $(1, 0)$ is on the graph of $f^{-1}(x) = \ln(x)$.
3. The domain of $f(x) = e^x$ is the range of $f^{-1}(x) = \ln(x)$: $-\infty < x < \infty$.
   The range of $f(x) = e^x$ is the domain of $f^{-1}(x) = \ln(x)$: $0 < x < \infty$.

   Note also that while the graph of $f(x) = e^x$ has a horizontal asymptote at $y = 0$,
   the graph of $f^{-1}(x) = \ln(x)$ has a vertical asymptote at $x = 0$.
4. $f \circ f^{-1} = e^{\ln x} = x$ and $f^{-1} \circ f = \ln(e^x) = x$.
5. The graphs of $f(x) = e^x$ and $f^{-1}(x) = \ln(x)$ are symmetric with respect to the line $y = x$. 
Use the reference point \( \log_b(1) = 0 \) and knowledge of the basic shape to graph simple logarithm functions using shifts and translations.

Examples D. Rewrite the following exponentials in logarithm form.

a. \( 5^x = 125 \)  

b. \( 13 = e^x \)

Examples E. Rewrite the following logarithms in exponential form.

a. \( \log_b 36 = 2 \)  

b. \( \ln x = 7 \)

Example F: Simplify \( \ln \left( \sqrt[5]{e^3} \right) \). Answer: \( \frac{3}{5} \)

Example G: Simplify \( e^{\ln(x+2)} \). Answer: \( x + 2 \)

Example H: Simplify \( e^{\ln(x) + 2} \). Answer: \( xe^2 \)

Example I: Solve \( 5 \ln(x - 1) + 4 = 0 \). Answer: \( e^{-\frac{4}{5}} + 1 \)

Example J: Solve \( \ln(x^2 - 1) = 4 \). Answer: \( \pm \sqrt{e^4 + 1} \)

Example K: Solve \( 5 - 2e^{-2x} = 0 \). Answer: \( -\frac{1}{2} \ln \frac{5}{2} \)