Calculus 130, section 3.1b Limits, section 3.2 Continuity
notes prepared by Tim Pilachowski

1) The notation \( \lim_{x \to a^+} f(x) \) is read “the limit of \( f(x) \) as \( x \) approaches \( a \) from the right”.

2) The notation \( \lim_{x \to a^-} f(x) \) is read “the limit of \( f(x) \) as \( x \) approaches \( a \) from the left”.

3) If there is some identifiable real number \( L \) such that \( \lim_{x \to a^+} f(x) = L \) and also \( \lim_{x \to a^-} f(x) = L \), then we will be able to write \( \lim_{x \to a} f(x) = L \), which is read “the limit of \( f(x) \) as \( x \) approaches \( a \)”. It is implicit in this statement that the limit is being taken as “\( x \) approaches \( a \) from either side”.

In some of the examples done last time, we had \( \lim_{x \to a} f(x) = \infty \). Now we turn to consideration of \( \lim_{x \to \infty} f(x) \).

(This is the concept underlying the determination of horizontal and slant asymptotes for rational and exponential functions.)

Examples D: For the following functions, determine whether \( \lim_{x \to \infty} f(x) = L \) exists, and if so, the value of \( L \).

\[
\begin{align*}
\text{a)} f(x) &= 3 & \text{b)} f(x) &= \frac{3x^2 - 1}{2} & \text{c)} f(x) &= \frac{3x^2 - 1}{2x - 1} & \text{d)} f(x) &= \frac{3x^2 - 1}{2x^2 - 1} & \text{e)} f(x) &= \frac{3x^2 - 1}{2x^3 - 1}
\end{align*}
\]

answers: 3, \( \infty \), \( \infty \), \( \frac{3}{2} \), 0
Examples E: For the following functions, determine whether \( \lim_{x \to \infty} f(x) = L \) exists, and if so, the value of \( L \).

a) \( f(x) = \ln(x - 1) \)  
   b) \( f(x) = e^{x-1} \)  
   c) \( f(x) = e^{1-x} \)  
   d) \( f(x) = \frac{3}{2 + e^{-x}} \)  
   e) \( f(x) = \cos(x - 1) \)

answers: \( \infty, \infty, 0, \frac{3}{2}, \text{DNE} \)

A pair of basic but very important principles are illustrated in the Examples above: As a denominator \( \to \infty \), a fraction \( \to 0 \), while any constants \( \to \) themselves.

With respect to power functions, for any real number \( n > 0 \), \( \lim_{x \to \infty} x^n = \infty \) and \( \lim_{x \to \infty} \frac{1}{x^n} = 0 \).

With respect to exponential functions, for any real number \( k > 0 \), \( \lim_{x \to \infty} b^{kx} = \infty \) and \( \lim_{x \to \infty} \frac{1}{b^{kx}} = 0 \).

Finally, for a given constant \( a \), \( \lim_{x \to \infty} a = a \).
An extension of limits is the concept of **continuity of a function**. Specifically, a function is continuous at a value \( x = c \) if and only if

1) \( f(c) \) is defined  
2) \( \lim_{{x \to c}} f(x) \) exists  
3) \( \lim_{{x \to c}} f(x) = f(c) \).

If a function is not continuous at \( x = c \), then it is **discontinuous**.

Examples A revisited: Determine whether the following functions are continuous at \( x = 1 \). Explain why.

\[ a) f(x) = x - 1 \quad b) f(x) = e^{x-1} \quad c) f(x) = \ln(x-1) \quad d) f(x) = \cos(x-1) \]

*answers: yes, yes, no, yes*

Examples B revisited: Determine whether the following functions are continuous at \( x = 1 \). Explain why.

\[ a) f(x) = \frac{1}{x-1} \quad b) f(x) = \frac{x^2 - 1}{x-1} \quad c) f(x) = \frac{x-1}{x^2 - 1} \quad d) f(x) = \frac{x-1}{x^2 + 1} \quad e) f(x) = \frac{|x-1|}{x-1} \]

*answers: no, no, no, yes, no*
Definition: A function is **continuous on an open interval** \((a, b)\) [i.e. \(a < x < b\)] if it is continuous at every \(x\)-value in that interval.

Examples B again: Determine all open intervals on which each of the following functions is continuous.

a) \(f(x) = \frac{1}{x-1}\)  
   ![Graph of f(x) = 1/(x-1)](image1)  
   **Answers:** \((-\infty, 1) \cup (1, \infty)\)

b) \(f(x) = \frac{x^2 - 1}{x-1}\)  
   ![Graph of f(x) = (x^2 - 1)/(x-1)](image2)  
   **Answers:** \((-\infty, 1) \cup (1, \infty)\)

c) \(f(x) = \frac{x-1}{x^2-1}\)  
   ![Graph of f(x) = (x-1)/(x^2-1)](image3)  
   **Answers:** \((-\infty, -1) \cup (-1, 1) \cup (1, \infty)\)

d) \(f(x) = \frac{x-1}{x^2+1}\)  
   ![Graph of f(x) = (x-1)/(x^2+1)](image4)  
   **Answers:** \((-\infty, 1) \cup (1, \infty)\)

e) \(f(x) = \frac{x-1}{x-1}\)  
   ![Graph of f(x) = (x-1)/(x-1)](image5)  
   **Answers:** \((-\infty, 1) \cup (1, \infty)\)

Examples F: Determine all intervals on which each of the following functions is continuous.

a) \(f(x) = \sqrt{x-1}\)  
   ![Graph of f(x) = sqrt(x-1)](image6)  

b) \(f(x) = \sqrt{9-x^2}\)  
   ![Graph of f(x) = sqrt(9-x^2)](image7)

Neither of these functions satisfies the definition of “continuous” at their endpoints: for a) at \((1, 0)\), and for b) at \((-3, 0)\) and \((3, 0)\). Is there a way that we can think about continuity so that we can include such endpoints?

Definition: A function is **continuous on a closed interval** \([a, b]\) if and only if

1) it is continuous on \((a, b)\)
2) it is continuous from the right at \(x = a\), i.e. \(\lim_{x \to a^+} f(x) = f(a)\)
3) it is continuous from the left at \(x = b\), i.e. \(\lim_{x \to b^-} f(x) = f(b)\)

Examples F: Determine all intervals on which each of the following functions is continuous.

a) \(f(x) = \sqrt{x-1}\)  
   ![Graph of f(x) = sqrt(x-1)](image6)  

b) \(f(x) = \sqrt{9-x^2}\)  
   ![Graph of f(x) = sqrt(9-x^2)](image7)

**Answers:** \([1, \infty)\), \([-3, 3]\)