You’ll need everything we covered in sections 2.1 – 2.3, especially exponential and logarithm properties.

Functions with the basic form \( y = b^x \) are actually a family of functions. Consider the functions \( y = 10^x, \ y = 5^x, \ y = 3^x, \ y = 2^x, \ y = 1.1^x \) pictured in the graph to the right. Note first the similarities: \( b^0 = 1 \) for all values of \( b \neq 0 \), so (0, 1) makes a good reference point. Each of the basic exponential functions has a horizontal asymptote \( y = 0 \). The graphs also have similar shape—the major difference is slope of the curve at specific values of \( x \). Note that at \( x = 0 \) slope of \( y = 10^x \) is steepest; slope of \( y = 1.1^x \) is most shallow. Our determination of first derivative will have to reflect this.

Recall that our limit definition for the first derivative, \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \), is akin to considering a series of secant lines where the second point comes ever closer to a fixed point \((x, f(x))\).

First, consider \( y = 5^x \) at \( x = 0 \) where \( y = 1 \). The slope of the secant line connecting \((0, 1)\) to another point on the curve is given by the formula \( \frac{5^{0+h} - 5^0}{h} = \frac{5h - 1}{h} \). As our second point approaches \((0, 1)\), the slope of the secant line approaches the slope of the tangent line, i.e slope of the tangent = \( \lim_{h \to 0} \frac{5^h - 1}{h} \). The table below provides results from successively smaller values of \( h \).

<table>
<thead>
<tr>
<th>( 5^0 = 1 )</th>
<th>( h = )</th>
<th>( 5^h )</th>
<th>( 5^h - 1 )</th>
<th>( \frac{5^h - 1}{h} )</th>
</tr>
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<td>1.000000016</td>
<td>0.000000016</td>
<td>1.609437916</td>
</tr>
</tbody>
</table>

The slope of the tangent to \( y = 5^x \) at \( x = 0 \), and therefore the first derivative of \( y = 5^x \) at \( x = 0 \), is approximately 1.61. This is an estimate and is not exact—with some higher-powered mathematics, we would identify the exact expression of the first derivative of \( y = 5^x \) at \( x = 0 \) as \( \ln(5) \).

Next we move to an arbitrary point \((x, 5^x)\) on the graph of \( y = 5^x \). The slope of the secant line connecting \((x, 5^x)\) to another point on the curve is given by the formula \( \frac{5^{x+h} - 5^x}{h} \), which can be simplified using the properties of exponents: \( \frac{5^{x+h} - 5^x}{h} = \frac{5^x * 5^h - 5^x}{h} = 5^x \left( \frac{5^h - 1}{h} \right) \). As our second point approaches \((x, 5^x)\), the slope of the secant line approaches the slope of the tangent line, i.e slope of the tangent = 

\[
\lim_{h \to 0} \frac{5^x (5^h - 1)}{h} = \left[ \lim_{h \to 0} \frac{5^h - 1}{h} \right] * 5^x = \left[ (5^x)' \right]_{x=0} * 5^x = \left[ \frac{d}{dx} \left( 5^x \right) \right]_{x=0} * 5^x \equiv \ln(5) * 5^x.
\]
If we rigorously followed a similar process for a generic function $y = b^x$, we would get $y' = \ln(b) \cdot b^x$.

When considering the natural exponential function $f(x) = e^x$, this gives us $f'(x) = \ln(e) \cdot e^x = e^x$! In short, the natural exponential function, $y = e^x$, is special because, unlike other functions, it is its own derivative. This property makes it not only very interesting, but also very useful.

Example A: Find the first derivative of $f(x) = \frac{x^3}{e^x}$ and solve $f'(x) = 0$. Answers: $\frac{3x^2 - x^3}{e^x}$; $x = 0, 3$

Example B: Given $h(x) = e^{x^2-x}$, find the first derivative. Answer: $\left(e^{x^2-x}\right)(2x-1)$

Example C: Let $g(t) = e^{2t} + e^{-2t}$. Determine where the graph of $g$ has horizontal tangents. Answer: $t = 0$

Example D: The accumulated amount of an investment of $100 with 5\% annual interest compounded continuously for $t$ years is given by the formula $A = 100e^{0.05t}$. Find and interpret the first derivative. Answer: $5e^{0.05t}$

From Examples C and D we can make a generic observation: Given an exponential growth/decay function $y = Ce^{kx}$, the derivative will be $y' = Cke^{kx} = k \cdot Ce^{kx} = ky$.

**Warning! Danger! Be careful!** This observation applies only to basic exponential growth and decay! It will not apply to other functions involving exponentials.
Example E: The exponential growth model $y = Ce^{kx}$ applied to populations of people or animals has a serious flaw: In the real world the number that can survive is limited by the amount of space and the number of resources available. A logistic growth curve is more appropriate for long-term applications.

The population of deer in a wildlife preserve is modeled by $P(t) = \frac{200}{3 + 5e^{-0.1t}}$. a) What was the number of deer at the beginning? b) What is the theoretical “upper limit” according to this model? c) How quickly is the population growing after 10 years? Answers: 25 deer; 66-67 deer; $\frac{100}{e^{(3+5/6)}} \approx 1.6$ deer per year

Example F: Given $f(x) = \pi^x$, find $f'(x)$. Answer: $(\ln \pi) \pi^x$

The natural logarithm function, $y = \ln(x)$, is the inverse of the natural exponential function, $y = e^x$. By finding the first derivative, we can determine that the slope of $y = e^x$ is 1 at the point $(0, 1)$, i.e. $\frac{d}{dx}(e^x) \bigg|_{x=0} = 1$. By symmetry, the slope of $y = \ln(x)$ should also be 1 at the point $(1, 0)$, i.e. $\frac{d}{dx}(\ln x) \bigg|_{x=1} = 1$. Also recall that $y = \ln(x)$ is increasing over its entire domain. The formula we use for the derivative of $\ln(x)$ must meet these conditions.
Finding a derivative formula for \( \ln(x) \) is actually quite simple. First note that since \( e^{\ln x} = x \), then
\[
\frac{d}{dx} \left( e^{\ln x} \right) = \frac{d}{dx} (x) = 1.
\]
By the chain rule,
\[
\frac{d}{dx} \left( e^{\ln x} \right) = e^{\ln x} \cdot \frac{d}{dx} (\ln x) = x \cdot \frac{d}{dx} (\ln x) = 1 = \frac{1}{x}.
\]

Note that \( \frac{d}{dx} (\ln x) \bigg|_{x=1} = \frac{1}{x} \bigg|_{x=1} = 1 \). Also \( \frac{d}{dx} (\ln x) = \frac{1}{x} > 0 \) and \( \frac{d^2}{dx^2} (\ln x) = -\frac{1}{x^2} < 0 \) for all \( x \) in the domain of \( \ln(x) \). In other words, all of the necessary conditions listed above have been met.

Also note that, since the domain of \( f(x) = \ln x \) is \((0, \infty)\), the domain of \( f'(x) = \frac{1}{x} \) is also \((0, \infty)\). What if we were to consider \( g(x) = \ln|x| \) which has a domain \( (-\infty, 0) \cup (0, \infty) \). Using symmetry of the graph of \( g \) across the \( y \)-axis, we could show that it is also true that \( g'(x) = \frac{1}{x} \) for all values of \( x \) in the domain \( (-\infty, 0) \cup (0, \infty) \).

Example G: Given \( h(x) = x^3 \cdot \ln|x| \) find the first derivative.

*Answers: \( x^2 \left( 1 + 3 \ln|x| \right) \)*

Example H: Given \( f(x) = \frac{x^3}{\ln|x|} \) find the first derivative.  
*Answer: \( \frac{x^2 \left( 3 \ln|x| - 1 \right)}{[\ln|x|]^2} \)*

Example I: Given \( g(x) = \frac{\ln|x|}{x^3} \) find the first derivative.  
*Answer: \( \frac{1 - 3 \ln|x|}{x^4} \)*

Carefully note the placement of coefficients when finding derivatives.
Example J: Given \( h(x) = \log_{\pi} (x^3) \), find the first derivative.  
\[ \text{Answer: } \frac{3}{\ln(\pi)} \cdot \frac{1}{x} \]

When using the chain rule, it is extremely important to correctly identify the “outside” and “inside” functions. Check that your composition is set up correctly.

Example K: Given \( h(x) = (\log_{\pi} x)^5 \), find the first derivative.  
\[ \text{Answer: } \frac{5(\ln x)^4}{\ln^5 \pi} \cdot x \]