Calculus 130, section 6.3 Implicit Differentiation
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All of the equations encountered so far have been functions, \( y = f(x) \): for example, \( y = 45t^2 - t^3 \) and \( P(t) = \frac{80}{2 + 3e^{-10t}} \). This is an *explicit* statement of the function formula, and given an explicit function and a value for \( x \), the determination of the corresponding \( y \)-coordinate becomes a calculation. In addition, the determination of the slope of the curve at that value of \( x \) means using one of the derivative rules developed so far, then calculating.

Even when a function is not expressed explicitly, it is sometimes possible to solve for the explicit version:

\[
5x + 2y = 12 \quad \Rightarrow \quad y = f(x) = -\frac{5}{2}x + 6.
\]

However, not all equations involving \( x \) and \( y \) can easily be rearranged algebraically into an explicit version, and others cannot be written explicitly at all. We’ll call these *implicit* equations. (See Examples below).

It is sometimes possible to find a derivative \( \frac{dy}{dx} \) from an implicit equation. The process is called *implicit differentiation*.

Example A: Given the equation \( 5x^2 + 2y^2 = 53 \), a) Verify that the point \((x, y) = (-3, 2)\) satisfies the equation. 

b) Use implicit differentiation to find \( \frac{dy}{dx} \). c) Find the equation of the tangent to the curve at \((x, y) = (-3, 2)\).

*answers:* \(-\frac{5x}{2y} \cdot \frac{15}{4} x + \frac{53}{4}\)

Example B: Given the equation \( \ln(x - y) = xy \), a) Use implicit differentiation to find \( \frac{dy}{dx} \). b) Find the equation of the tangent to the curve at \((x, y) = (1, 0)\).

*answers:* \( \frac{xy - y^2 - 1}{1 - x^2 + xy} \cdot \frac{1}{2} x - \frac{1}{2} \)
Example C: Given the equation \( x^2 y^3 = 1 \), a) Use implicit differentiation to find \( \frac{d^2 y}{dx^2} \). b) Solve for the explicit equation and find \( \frac{d^2 y}{dx^2} \). c) Show that the results from (a) and (b) are equal.  
\text{answer:} \frac{10y}{9x^2}

Example D: Given the equation \( 2x + 3y = e^{\sin(xy)} \), find \( \frac{dy}{dx} \).  
\text{answer:} \frac{ye^{\sin(xy)} \cos(xy) - 2}{3xe^{\sin(xy)} \cos(xy)}

Example E. Given the equation \( u^3 - uv + 2v^2 = 0 \), use implicit differentiation to find a) \( \frac{du}{dv} \) and b) \( \frac{dv}{du} \).  
\text{answers:} \frac{u - 4v}{3u^2 - v}, \frac{v - 3u^2}{4v - u}