Math 130 Exam 3 Sample 2 Solutions

1. First we take the derivative and clean it up:

\[ f(x) = (x - 3)(x - 1)^3 \]
\[ f'(x) = (1)(x - 1)^3 + (x - 3)(x - 1)^2 \]
\[ f''(x) = (x - 1)^2 \left[(x - 1) + 3(x - 3)\right] \]
\[ f''(x) = (x - 1)^2(4x - 10) \]

This function is never undefined and equals 0 when \( x = 1 \) and \( x = 5/2 \). We test these and the endpoints:

\[ f(1) = (-2)(0)^3 = 0 \]
\[ f(5/2) = (-1/2)(3/2)^3 = -27/16 \]
\[ f(-2) = (-5)(-3)^3 = 135 \]
\[ f(3) = (0)(2)^3 = 0 \]

The maximum is then 135 and the minimum is \(-27/16\).

2. The perimeter of the rectangle is \( 2x + 2y \), however \( y = \frac{3}{2} \) and so the perimeter is \( P(x) = 2x + 2\left(\frac{3}{2}\right) = 2x + 3 \). This is valid for \( x > 0 \) (the first quadrant).

We have \( P'(x) = 2 - \frac{6}{x^2} \) which is undefined at \( x = 0 \) (but this is not in the interval \( x > 0 \)) so we set it equal to 0:

\[ 2 - \frac{6}{x^2} = 0 \]
\[ 2x^2 - 6 = 0 \]
\[ x^2 = 3 \]
\[ x = \pm \sqrt{3} \]

We throw out \(-\sqrt{3}\) because it’s not in the interval. Thus we have \( x = \sqrt{3} \).

To see that it’s a minimum note that a number line sketch for \( P'(x) \) shows us:

\[ \sqrt{3} \quad P'(x) = - \quad P(5) = + \]

From which we see that \( P(x) \) is at its lowest point at \( x = \sqrt{3} \).
3. (a) We differentiate implicitly:

\[ x^2 y^2 - \frac{x}{y} = 3x \]

\[ \frac{d}{dx} \left[ x^2 y^2 - \frac{x}{y} \right] = \frac{d}{dx} [3x] \]

\[ 2xy^2 + x^2 y \frac{dy}{dx} - \frac{1(y) - x(dy/dx)}{y^2} = 3 \]

\[ 2xy^2 + x^2 y \frac{dy}{dx} - \frac{1}{y} + \frac{x dy}{y^2 dx} = 3 \]

\[ \frac{dy}{dx} \left[ 2x^2 y + \frac{x}{y^2} \right] = 3 - 2xy^2 + \frac{1}{y} \]

\[ \frac{dy}{dx} = \frac{3 - 2xy^2 + 1/y}{2x^2 y + x/y^2} \]

At (2, −1) this slope is

\[ \frac{dy}{dx} = \frac{3 - 2(2)(-1)^2 + 1/(-1)}{2(2)^2(-1) + 2/(-1)^2} = \frac{-2}{-6} = \frac{1}{3} \]

So the line has equation \( y - (-1) = \frac{1}{3}(x - 2) \).

(b) We are given \( \frac{dA}{dt} = 5 \) and we want \( \frac{dr}{dt} \). This means we need an equation relating \( A \) and \( r \).

The area of a circle is \( A = \pi r^2 \), this does the job. We implicitly differentiate with respect to \( t \) and get

\[ \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \]

Now then, when \( A = 100 \) we have \( 100 = \pi r^2 \) and so \( r = \frac{10}{\sqrt{\pi}} \) and so we plug \( \frac{dA}{dt} \) and \( r \) into our equation:

\[ \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \]

\[ 5 = 2\pi \left( \frac{10}{\sqrt{\pi}} \right) \frac{dr}{dt} \]

\[ 5 = 20 \sqrt{\pi} \frac{dr}{dt} \]

\[ \frac{dr}{dt} = \frac{1}{4\sqrt{\pi}} \]

The units are inches per year.
4. (a) \( \int \frac{1}{x} - 2x + 3 \, dx = \ln |x| - x^2 + 3x + C \)

(b) \( \int x(x + 2)(5x - 1) \, dx = \int x^3 + 9x^2 - 2x \, dx = \frac{1}{4}x^4 + 3x^3 - x^2 + C \)

(c) \( \int \frac{x^2 + 3x + 5}{x^2} \, dx = \int \frac{x^2 + 3x + \frac{5}{x}}{x^2} \, dx = \int e^x - \frac{3}{x} + x^3 \, dx = e^x - 3\ln |x| + \frac{1}{4}x^4 + C \)

(d) \( \int 2^{5x} \, dx = \frac{1}{\ln 2} \cdot 2^{5x} + C \)

5. (a) We let 
\[
\begin{align*}
u &= 1 + 4\cos x \\
du &= -4\sin x \\
du &= -4\sin x \, dx \\
\frac{1}{4} \, du &= \sin x \, dx
\end{align*}
\]

So then
\[
\int \sin x e^{(1+4\cos x)} \, dx = \int e^u \left( -\frac{1}{4} \, du \right) = \int -\frac{1}{4}e^u \, du = -\frac{1}{4}e^u + C = -\frac{1}{4}e^{(1+4\cos x)} + C
\]

(b) We let 
\[
\begin{align*}u &= 4x - 8 \\
du &= 4 \, dx \\
\frac{1}{4} \, du &= dx
\end{align*}
\]

Note we also need \( x = \frac{u+8}{4} \) here. So then
\[
\int \frac{x^2}{4x - 8} \, dx = \int \frac{(u+8)/4)^2}{u} \cdot \frac{1}{4} \, du
\]
\[
= \int \frac{1}{64} \cdot \frac{u^2 + 16u + 64}{u} \, du
\]
\[
= \int \frac{1}{64} \left[ u + 16 + \frac{64}{u} \right] \, du
\]
\[
= \frac{1}{64} \left[ \frac{1}{2}u^2 + 16u + 64 \ln |u| \right] + C
\]
\[
= \frac{1}{64} \left[ \frac{1}{2}(4x - 8)^2 + 16(4x - 8) + 64 \ln |4x - 8| \right] + C
\]
(c) Since the rate \( D'(t) \) is given we have \( D'(t) = 4t + 5 \). We find the antiderivative

\[
D(t) = \int D'(t) \, dt = \int (4t + 5) \, dt = 2t^2 + 5t + C
\]

Thus \( D(t) = 2t^2 + 5t + C \). Now then we know \( D(12) = 100 \) and so

\[
2(12)^2 + 5(12) + C = 100
\]

\[
288 + 60 + C = 100
\]

\[
C = -248
\]

Thus

\[
D(t) = 2t^2 + 5t - 248
\]

Then after two years there are \( D(24) = 2(24)^2 + 5(24) - 248 \) people with the disease.