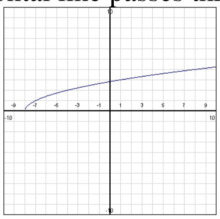
Math 115 Chapter 2.8 Examples and Extra Notes

Inverse of a function

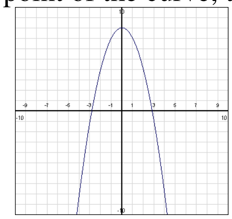
You can tell whether on not a function has an inverse by applying the “horizontal line test” to the graph: If any horizontal line passes through only one point of the curve, then the function has an inverse.



$y=\sqrt{x+8}$

passes the horizontal line test

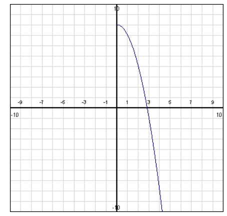
has an inverse



$y=-x^2+8$

doesn’t pass the horizontal line test

doesn’t have an inverse



$y=-x^2+8, x\ge 0$

passes the horizontal line test

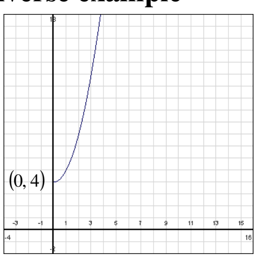
only because domain is restricted

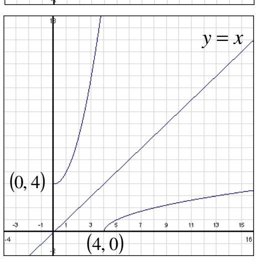
has an inverse

Given a function f and its inverse, f^{-1}, the following will always be true:

1. If $f(a)=b$, then $f^{-1}(b)=a$. (This fact and the statement in point #2 below is actually the same information.)
2. If (a, b) is a point on the graph of f, then (b, a) will be on the graph of f^{-1}. You can use this fact to use a graph to sketch its inverse: locate two or three points on the graph of *f*, swap coordinates, plot the new points, connect the dots, and you have the graph of f^{-1}. (See example below.)
3. The domain of *f* = the range of f^{-1}, and the range of *f* = the domain of f^{-1}.
4. The graph of *f* and the graph of f^{-1} are symmetric with respect to the line $y=x$.
5. $f\circ f^{-1}=x$ and $f^{-1}\circ f=x$. To show that two functions are inverses, you must do *both* compositions.

Inverse example





$f(x)=x^2+4,\ x\ge 0$, passes the horizontal line test (but only because the domain is restricted) so it has an inverse. Using #1 and #2 from above, since the points $(0, 4)$ and $(2, 8)$ are on the graph of *f*, then $(4, 0)$ and $(8, 2)$ will be on the graph of $f^{-1}(x)$.

The graph of f^{-1} has been added in the second picture to the left. Item #4 above is verified, since the two graphs are symmetric across the line $y=x$.

The domain of *f* has been restricted to $[0,\infty)$. From the graph we can tell that its range is $[4, \infty)$. Likewise, using the graph of f^{-1} we can see that its domain is $[4, \infty)$, while its range is $[0,\infty)$. That is, the domain of *f* = the range of f^{-1}, and the range of *f* = the domain of f^{-1} (#3 above).

Using what we know about transformations of graphs, it appears that the equation of f^{-1} is $f^{-1}(x)=\sqrt{x-4}$. We can verify this using item #5:

$f\circ f^{-1}=f\left(\sqrt{x-4}\right)=\left(\sqrt{x-4}\right)^2+4=x-4+4=x$ $\qquad$ $f^{-1}\circ f=f^{-1}\left(x^2+4\right)=\sqrt{\left(x^2+4\right)-4}=\sqrt{x^2}=|x|=x$

Important note: Because the domain of *f*, and therefore $f^{-1}\circ f$, is restricted to positive numbers, and absolute value of a positive number is the number itself, for *these two functions* the composition $f\circ f^{-1}$ will *always* equal *x*.

Finding the inverse of a function

Given a function such as $m(x)=3x+5$, we know it has an inverse since it is a line which passes the horizontal line test. Using items #1 through #3 above, we can develop a process to find the inverse of $m(x)$: If we switch *x* and *y* in the equation we transform $m(x)$ into its inverse. The process looks like this:

$m(x)=3x+5 \rightarrow y=3x+5$

switch *x* and *y*: $x=3y+5$

solve for *y*: $-3y+x=5$

$$-3y=-x+5$$

$$y=\frac{-x+5}{-3}=\frac{1}{3}x-\frac{5}{3}$$

$$m^{-1}(x)=\frac{1}{3}x-\frac{5}{3}$$

We can verify that we have the correct equation for the inverse by using composition (item #5 above).

$$m\circ m^{-1}=m\left(\frac{1}{3}x-\frac{5}{3}\right)=3\left(\frac{1}{3}x-\frac{5}{3}\right)+5=x-5+5=x$$

$$m^{-1}\circ m=m^{-1}(3x+5)=\frac{1}{3}(3x+5)-\frac{5}{3}=x+\frac{5}{3}-\frac{5}{3}=x$$