Precalculus 115, section 4.4 Logarithm Properties

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Recall the properties of exponential functions:

$$b^x * b^y = b^{x+y} \qquad \frac{b^x}{b^y} = b^{x-y} \qquad \frac{1}{b^y} = b^{0-y} = b^{-y} \qquad \left(b^x\right)^y = b^{xy} \qquad a^x * b^x = (ab)^x \qquad \frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x.$$

Logarithm functions, and in particular the *natural logarithm* function, $y = \ln x$, have properties of their own, related to but not exactly the same as the exponential function properties. The logarithm properties apply to logarithms in any base, including $\ln(x)$. Of the three properties listed in the text, two are vital. The other one can be easily derived from these.

$$\log_b(AB) = \log_b(A) + \log_b(B) \qquad \log_b\left(A^C\right) = C\log_b(A)$$

To prove the first: Note that $e^{\ln(AB)} = AB$. But since $A = e^{\ln A}$ and $B = e^{\ln B}$, we can use the properties of exponents to write $e^{\ln(AB)} = A * B = e^{\ln A} * e^{\ln B} = e^{\ln A + \ln B}$. Since the exponents must be equal, we're done. [We used $\ln(AB)$. This proof as well as the one below can be easily duplicated for logarithms in any base b.]

Proof of the second vital property is along similar lines: $e^{\ln\left(A^C\right)} = A^C = \left(e^{\ln A}\right)^C = e^{C * \ln A}$.

Knowing that $\frac{1}{B} = B^{-1}$, the other property given by the text becomes fairly obvious:

$$\ln\frac{A}{B} = \ln\left(AB^{-1}\right) = \ln A + \ln\left(B^{-1}\right) = \ln A - \ln B.$$

Example A: Simplify $\ln 60 - \ln 4 - \ln 5$.

Example B: Evaluate $\log_5\left(25^{97}\right)$.

Example C: Evaluate $\log_3\left[3\ln\left(e^9\right)\right]$.

Example D: Simplify $e^{3\ln 5 - \frac{1}{3}\ln 8}$.

Example E: Using properties of logarithms, expand and simplify $\log_5\left(\frac{125a^3}{b^4\sqrt[5]{c}}\right)$.

Example F: Using properties of logarithms, expand and simplify $\ln\left(x^2e^x\right)$.

Example G: Using properties of logarithms, expand $\log_3\left[\frac{3^{x+1}}{(3x+7)^5\sqrt{x^2-5}}\right]$.

Be careful when using the logarithm properties! They *do not* allow us to expand this example any more—the properties apply when a product or quotient is inside the logarithm, but *not* when there is a sum or difference.

One more logarithm property: the change of base formula: $\log_b(x)=\frac{\ln x}{\ln b}$.

To prove the change of base formula: Note that $y=\log_b x$ can be written in its exponential form $b^y=x$.

Taking ln of both sides gives us $\ln\left(b^y\right)=\ln x \quad \Rightarrow \quad y\ln b=\ln x \quad \Rightarrow \quad y=\frac{\ln x}{\ln b} \ \left(=\log_b x\right)$.

For this class, we'll use the change of base formula mostly to approximate the placement of *y*- and *x*-intercepts.

Example H: Use the change of base formula to approximate $\log_7(30)$.