Calculus 120, section 1.6 Some Basic Properties of Derivatives

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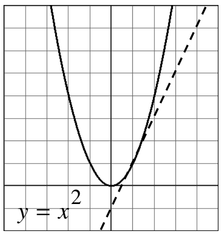
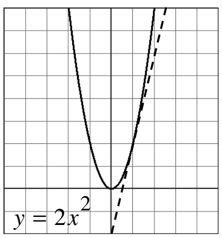
Today we take the ideas of section 1.3 (the first derivative) and expand a little bit.

Given a function $y = k * f(x)$, where k is a constant coefficient of a variable function, what would be its first derivative? Think in terms of what you know about transformations and what you've learned about slope of the tangent line. The constant will either stretch the graph (when $|k| > 1$) or shrink the graph (when $|k| < 1$). What effect will this stretch/shrink have on the slope of the tangent line? Will it stretch/shrink at the same rate as the curve? Consider the quadratics pictured to the left. For $f(x) = x^2$, $f(1) = 1$ and the slope of the tangent line = 2. For $f(x) = 2x^2$, $f(1) = 2$ and the slope of the tangent line = 4. We might begin to suspect that as a function undergoes a stretch/shrink, the tangent line stretches/shrinks at the same rate. Indeed, if we were to do more with limits, it would be easy to show that





$$\lim_{h \to 0} \frac{kf(x+h) - kf(x)}{h} = \lim_{h \to 0} \frac{k[f(x+h) - f(x)]}{h} = k * \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

which is to say that given a function $y = k * f(x)$, $\frac{dy}{dx} = k * f'(x)$ [**constant-multiple rule**].

Example A: Given $f(x) = 7x^3$, find the derivative. *Answer*: $21x^2$

Can we do the same if we add two functions together? That is, Given $p(x) = f(x) + g(x)$, does $p'(x) = f'(x) + g'(x)$? Since $\lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$

$$= \lim_{h \to 0} \frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

, the answer is "Yes."

The derivative of a sum is the sum of the derivative [**sum rule**]. Note that the constant-multiple rule and sum rule work together to give us the derivative of a subtraction, since $f(x) - g(x) = f(x) + (-1 * g(x))$.

Example B: Given $f(x) = x^3 - 8x + 2$, find the first derivative. *Answer*: $3x^2 - 8$

Example C: Given $g(x) = 2x^5 - \frac{x^4}{4} + 3\sqrt[3]{x} - \frac{7}{x^2}$, find the first derivative. *Answer*: $10x^4 - x^3 + \frac{1}{x^{2/3}} + \frac{14}{x^3}$

Caution, Be careful, Warning, Warning! Danger, Will Robinson! There is no similar easy process for the derivative of a product, nor is there a similar easy process for the derivative of a quotient. We'll need to work a good bit for those. Until we derive them, however, there are still a few things we can do.

Example D: Given $m(x) = (4x^2 - 1)(3 + x^3)$, find m'. *Answer*: $20x^4 - 3x^2 + 24x$

Example E: Given $n(x) = (5x^4 - 1)^2$, find $n'(x)$. *Answer*: $200x^7 - 40x^3$

One more property (rule) to discuss. Your text calls it the **general power rule**. It's one version of something that, later on, we'll call the **chain rule**, which applies to differentiating a composition of functions:

$$\frac{d}{dx}\left([g(x)]^r\right) = r\left([g(x)]^{r-1}\right) * \frac{d}{dx}[g(x)]$$

(Although it's not a rigorous mathematical description, I tend to think of this as "the derivative of the outside applied to the inside times the derivative of the inside".)

Example E revisited: Given $n(x) = (5x^4 - 1)^2$, find $n'(x)$. *Answer*: $40x^3(5x^4 - 1)$

Example F (one more for the fish): Given $y = \left(3x^2 - \frac{2x+1}{x^2}\right)^9$, find $\frac{dy}{dx}$.

Answer: $9\left(3x^2 - \frac{2x+1}{x^2}\right)^8\left(6x + \frac{2}{x^2} + \frac{2}{x^3}\right)$

Note the clever way we manipulated the quotient into a sum of power functions, being careful to distribute the negative, and using exponent properties to simplify.