Calculus 120, section 1.7 More about Derivatives

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Section 1.7 adds some depth to our discussion of derivatives of functions. To review what we already have, the following statements are mathematically equivalent:

a) Find the slope of the line tangent to the graph of f at a point (x, y).

b) Find $\lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$.

c) Find the first derivative of $f(x)$.

d) Find $f'(x)$.

e) Find $\dfrac{dy}{dx}$.

Recall, however, that the first derivative is itself a function, which has its own domain and its own graph. Since it is a function, it also has its own derivative. Given a function f, we can calculate the first derivative f' or $\dfrac{dy}{dx}$.

We can then calculate the derivative of f', also called the second derivative of f, symbolically f'' or $\dfrac{d^2y}{dx^2}$.

<u>Important note</u>: Just like $\dfrac{dy}{dx}$ is *not* a fraction, but is a notation for the first derivative, $\dfrac{d^2y}{dx^2}$ is also not a fraction but a notation. *There is no multiplication involved!* Rather, you need to interpret it this way:

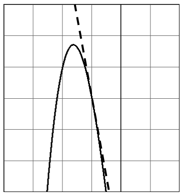
$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ which means "the derivative of $\dfrac{dy}{dx}$", the derivative of a derivative.

For now we'll focus on finding the first and second derivatives. In later sections we'll be exploring what these two tell us about the original function f.

Example A: Given $f(x) = x^3 - 8x + 2$, find the first and second derivatives. *Answers* $3x^2 - 8$; $6x$

Example B: Given $y = \left(5x^4 - 1\right)^2$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. *Answers*: $200x^7 - 40x^3$; $1400x^6 - 120x^2$

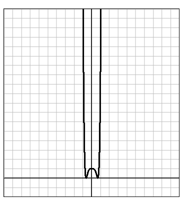
We can, of course, evaluate a function, a first derivative, or a second derivative at a specific point.



Example A revisited: Given $f(x) = x^3 - 8x + 2$, find $f(-1)$, $f'(-1)$, and $f''(-1)$.

The point $(-1, 9)$ is on the graph of *f*.
At the point $(-1, 9)$ the graph of *f* has a slope of -5.
At the point $(-1, 9)$, $f'' = -6$ and the curve is *concave down*.

Example B revisited: Given $y = (5x^4 - 1)^2$, find y when $x = -1$, $\left.\frac{dy}{dx}\right|_{x=-1}$ and $\left.\frac{d^2y}{dx^2}\right|_{x=-1}$.



With this example a new notation is introduced for evaluating a derivative at a specific point.

At the point $(-1, 16)$ the graph has a slope of -160.
At the point $(-1, 16)$, $f'' = 1280$ and the curve is *concave up*.

In summary (looking ahead to chapter 2):

To find a point on the curve, use the formula for $y = f(x)$.

To find the slope of the curve, use the formula for $y' = f'(x) = \frac{dy}{dx}$.

To find the concavity of the curve, use the formula for $y'' = f''(x) = \frac{d^2y}{dx^2}$

It is possible to find not only first and second derivatives, but also third, fourth, fifth, etc. Note that in the case of a polynomial, all higher order derivatives would equal 0 (specifically beginning with the $(n + 1)$st derivative of an nth degree polynomial.

Although in algebra we have traditionally, thanks to Descartes, used letters from the beginning of the alphabet (*a*, *b*, and *c*) and letters from the end of the alphabet (*x*, *y*, and *z*) to denote variables, it does not have to be so. In specific applications the letter is often chosen for ease of referring to what it represents. The notations for derivatives are adjusted accordingly.

Example C: The function $s(t) = -16t^2 + v_0 t + s_0$ calculates the height of an object, s, after time, t, thrown with an initial velocity v_0 and initial height s_0. Write a function for a rock thrown upward at 10 feet per second from a bridge which is 240 feet above the river below. Then find first and second derivatives: $s'(t)$ and $s''(t)$.
Answers: $-32t + 10$; -32

Foreshadowing: The first derivative in this situation describes velocity (in feet per second), and the second derivative describes acceleration (in feet per second per second).

In business and economics you will often see p used to represent price, x to represent demand [Why not use d?], C to represent cost, R to represent revenue, and P to represent profit. In this case, the notation for derivatives would be adjusted to fit the situation.

Example D: A small lampshade manufacturer has determined several equations which describe the economics of his company. The cost to produce x lampshades is $C(x) = \frac{1}{8}x^2 + 4x + 200$. Lampshades are sold for $p(x) = 79 - x$. Revenue is thus $R(x) = x * p(x) = x(79 - x) = 79x - x^2$. Write an equation to describe profit and find the first and second derivatives: $\frac{d}{dx}[P(x)]$ and $\frac{d^2}{dx^2}[P(x)]$. *Answers*: $-\frac{9}{8}x^2 + 75x - 200$; $-\frac{9}{4}x + 75$; $-\frac{9}{4}$

As a point of interest, which we'll deal with in more depth in a later section,

$C'(x)$ is called the *marginal cost*,

$R'(x)$ is called the *marginal revenue*, and

$P'(x)$ is called the *marginal profit*.

The maximum profit will be found where $P'(x) = R'(x) - C'(x) = 0$, that is, where $R'(x) = C'(x)$.