Calculus 120, section 1.8 The Derivative as a Rate of Change

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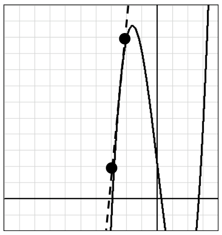
So far, most of our focus has been on derivatives of functions as the slope of the curve at a point. But recall the derivation of slope for a line as a rate of change: $m = \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x}$. So it's no real surprise that we can talk about the derivative of any function as a rate of change. Let's explore the concept a little.

Example A: Given $f(x) = x^3 - 8x + 2$, find the *average rate of change* over the interval $[-3, -2]$.

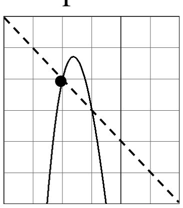


Answer: 11

Note two things: The "average rate of change" is the same as the slope of the secant line connecting the two points. Also, we have merely calculated the difference quotient, $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$.

Our calculated average rate of change is in between the slopes of the curve at the two points, i.e. $19 > 11 > 4$. It makes sense that this is so, since *f* is increasing over the entire interval, but will the same be true where *f* changes from increasing to decreasing?

Example B: Given $f(x) = x^3 - 8x + 2$, find the *average rate of change* over the interval $[-2, -1]$. *Answer*: –1



On the interval from –2 to –1, *f*(*x*) is increasing until it reaches its relative maximum (which we already know is at $x = -2\sqrt{\frac{2}{3}}$) and then decreases from there to $x = -1$. Another way to say this is to note that the slope of the curve is positive at $x = -2$ and negative at $x = -1$. Note that, once again the "average rate of change" is the same as the slope of the secant line connecting the two points. Also once again, our calculated average rate of change is in between the slopes of the curve at the two points, i.e. $4 > -1 > -5$.

By now you may be asking yourself, "If we already know about the first derivative, why should we even bother with the *average* rate of change?" There actually are several reasons. On the application side, we found the first derivative by using the equation. In cases where we have some information but not the equation, the average rate of change of two points very close together can give us a good estimate of the instantaneous rate of change, and vice versa.

Example C: When you start a job at a local restaurant, you earn $5.75 per hour. A year later you get a raise to $6.60 per hour. A year and a half after the first raise you get another, to $7.70 per hour. In which space of time did your earnings have a higher rate of change? *Answer*: during the first year

The average rate of change provides a way to compare raises given over differing periods of time.

Example D: Given, $f(20) = 100$, and $f'(20) = -2$, estimate $f(21)$, $f(20.1)$, and $f(20.01)$. *Answers*: 98; 99.8; 99.98

We used the fact that first derivative $\approx$ average rate of change $= \dfrac{\Delta f}{\Delta x}$.

Note again that we are calculating the difference quotient!

On the conceptual side, the average rate of change, which is the slope of the secant line connecting the two points, is what led us to the first derivative, which is the limit of the average rate of change as we move the two points closer together. In this sense, we can also think about the derivative as being a rate of change, the *instantaneous rate of change*. The first derivative is the slope of the curve at a given point, and is therefore an expression of how the curve is changing at that point, i.e. it is the rate of change of the function at that point. (On either side of the point, the slope of the curve/rate of change may be different.) Symbolically,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Example E: Find the rate of change of $f(x)=x^3-8x+2$ at $x=-2.1$, $x=-2$, and $x=-1.9$.
Answers: 5.23; 4; 2.83

The first derivative can be used to find rates of change in specific applications. Take, for example, the relationship between distance and velocity. Starting from the formula $d = rt$, we can derive $r=\frac{d}{t}$. The distance traveled is a change in position, y_2-y_1. Time traveled, like the time passing in Example C above is also a calculation of change, t_2-t_1. Velocity is the rate of change of position with respect to time, $\frac{\Delta y}{\Delta t}$. As you accelerate onto an interstate highway from 0 to 60 mph, at some particular moment in time your velocity is exactly 30 mph. It may be for seconds at a time, or it may be *instantaneous*, for that moment and no other. Instantaneous velocity is the first derivative of position with respect to time, $\frac{dy}{dt}$.

Example F: The function $s(t)=-16t^2+10t+240$ calculates the height of an object, s, after time, t, thrown upward at 10 feet per second from a bridge which is 240 feet above the river below. a) What is the height of the rock after 2 seconds? b) How much height did the rock gain after 2 seconds? c) What is the velocity of the rock after 2 seconds? d) What is the average velocity during the first 2 seconds? *Answers*: 196 ft; loss of 44 ft; falling at 54 fps; falling at 22 fps

Example G: Alfred's price function is $p(x)=8-0.001x$. Find the marginal revenue equation.
Answer: $8 - 0.002x$