Calculus 120, section 2.1 Describing Graphs of Functions

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Reminder: You will not be able to use a graphing calculator on tests!

First, a quick scan of what we know so far.

The slope of a curve at a point
= slope of line tangent to the curve at that point
= (instantaneous) rate of change of the curve at that point
= first derivative evaluated at that point.

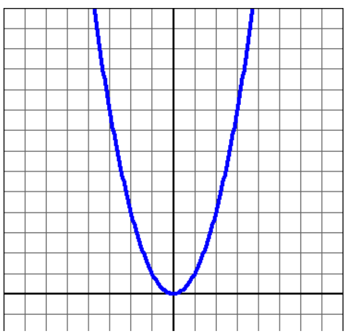
Notations are $f'(x)$ and $\frac{dy}{dx}$.

To find the first derivative of a given function we have
the power rule (both general and specific),
constant multiple rule, and
sum rule.

We also have that the derivative of the first derivative is the second derivative,

with notations $f''(x)$ and $\frac{d}{dx}\left[\frac{d}{dx}(f)\right] = \frac{d^2y}{dx^2}$.

Chapter 2 applies all this in various circumstances, and explores the meaning of it all.



Example A. Consider the graph of $y = x^2$ pictured to the left. Reading from left to right—

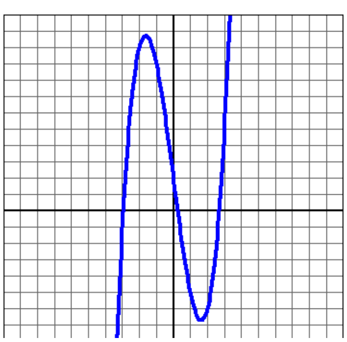
From "forever left" $(-\infty)$ to $x = 0$, the curve is going down = graph is falling = the slope of the curve is negative.

The graph "bottoms out" at the vertex (0, 0) where the slope of the curve = 0.

From $x = 0$ onward to "forever right" (∞) the curve is going up = the graph is rising = the slope of the curve is positive.

In technical terms, the function $f(x) = x^2$ is *decreasing* on the interval $-\infty < x < 0$, has a *minimum* at (0, 0), and is *increasing* on the interval $0 < x < \infty$. We can say that the *minimum value of f* is equal to 0 because there are no lower values in the range of *f*. In other words, the minimum here is an *absolute minimum*.

Example B. Let's go back to $f(x) = x^3 - 8x + 2$ and take a closer look at the curve, pictured to the left below.



increasing:

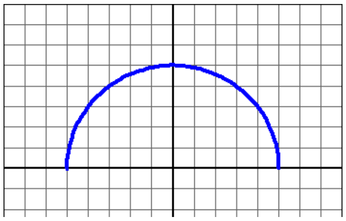
decreasing:

relative maximum:

relative minimum:

The function $f(x) = x^3 - 8x + 2$ has no absolute maximum or minimum; the range is $-\infty < x < \infty$. Vocabulary to know: *relative extrema* (plural) and *relative extremum* (singular).

Example C. The function $f(x) = \sqrt{25 - x^2}$ has a limited domain, $-5 \le x \le 5$, and range, $0 \le y \le 5$.

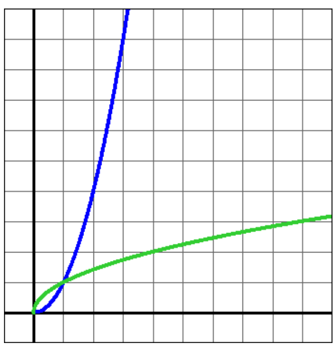


increasing:

absolute maximum:

decreasing:

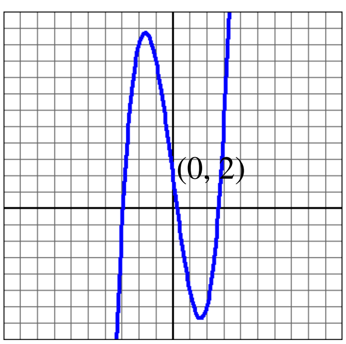
The minimum value of the function is 0. Because the minimum occurs at the endpoints of the domain it is called an *endpoint extreme value* or *endpoint extremum*. The maximum value of the function is 5.



Example D. Compare the graphs of $y = x^2$ and $y = \sqrt{x}$ on the interval $0 < x < \infty$. Neither has an absolute maximum. Also both are increasing on this domain, but in different ways.

$y = x^2$:

$y = \sqrt{x}$:

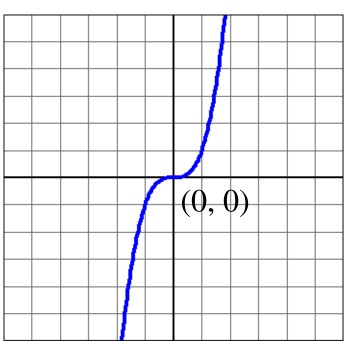


Example B revisited. Going back to $f(x) = x^3 - 8x + 2$ —

concave down:

concave up:

point of inflection:



Example E. Now consider the graph of $y = x^3$ —

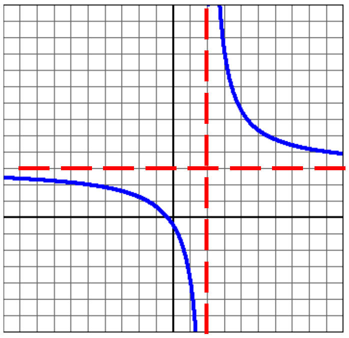
concave down:

concave up:

point of inflection:

Note also that the curve "levels out" at this spot, and that, at $x = 0$, the slope of the curve is 0. We'll say that this function is increasing for all values of x in the domain, $-\infty < x < \infty$.

Not all functions have unbounded domains. For example, $f(x)=\sqrt{25-x^2}$ is not defined for values of x less than –5 or greater than 5. The domain of $y=\sqrt{x}$ is $0 \le x < \infty$. Rational functions will usually have limited domains: they are not defined for values of x that would put a 0 in the denominator.



Example F. For example, as x approaches 2 from the left, the function $f(x)=\dfrac{3x+1}{x-2}$ approaches $-\infty$. As x approaches 2 from the right, the function approaches ∞.

vertical asymptote:

As x approaches both $-\infty$ and ∞, the value of f approaches 3.

horizontal asymptote:

Both x-intercepts and y-intercepts will also be of great interest and use to us in determining an accurate shape for the graph of a function.

In summary:

To find a point on the curve, use the formula for $y = f(x)$.

To find intervals on which the curve is increasing or decreasing, first find maxima and minima by setting $f'(x)=0$. The intervals will be in-between maxima and minima: increasing where $f'(x)>0$, and decreasing where $f'(x)<0$.

To find intervals on which the curve is concave up or concave down, first find points of inflection by setting $f''(x)=0$. The intervals will be on either side of points of inflection: concave up where $f''(x)>0$, and concave down where $f''(x)<0$.

Example G. Now its time to put all of this terminology to work in describing the graph of a new function: $f(x)=\dfrac{x^3}{e^x}$, whose graph is to the right. Numbers used correspond to the attributes listed in your text.

1) increasing/decreasing

2) maximum/minimum

3) points of inflection

4) x and y-intercepts

5) domain/undefined values

6) vertical/horizontal asymptotes