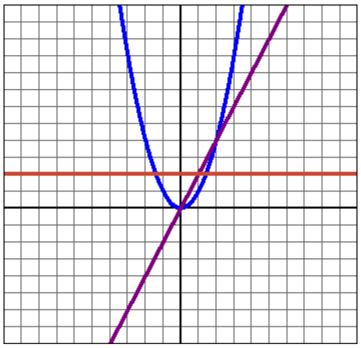
Calculus 120, section 2.2 First and Second Derivative Rules

notes by Tim Pilachowski

Last time, we did a visual review of graphs, looking at six items: increasing/decreasing, maximum/minimum (relative and absolute), inflection points and concave up/down, x and y-intercepts, points at which the function is undefined, and asymptotes. This time, we explore how first and second derivatives tell us about some of these attributes.



Consider the graph of $y = x^2$ pictured to the left along with its derivatives $y' = 2x$ and $y'' = 2$.

interval	$y = x^2$ is…	$y' = 2x$ is…	$y'' = 2$ is…

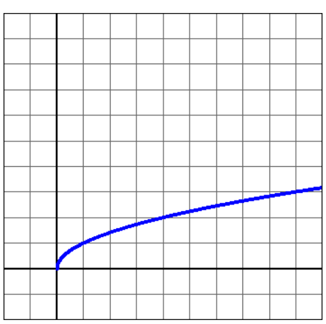
What is the connection between the concavity of a function and its second derivative? The second derivative is the slope of the first derivative, and tells us how the first derivative is changing, i.e. how the slope of the function is itself changing. In the graph of $y = x^2$ above, the slope (first derivative) is negative on the interval $-\infty < x < 0$. Note that the slope of the parabola is becoming less steep (more shallow) as x approaches 0. Another way to say the same thing is that the slope of the parabola (first derivative), while still negative, is becoming less negative as x approaches 0, until the curve hits $x = 0$, at which point the slope of the parabola (first derivative) equals 0. On the interval $0 < x < \infty$ the slope of the parabola (first derivative) is positive. Note, too, that the slope of the parabola is becoming steeper (i.e. the first derivative is becoming ever-larger positive numbers) as x approaches ∞. The slope of the curve = the first derivative is progressing in this way:

very negative < less negative < zero < small positive < large positive

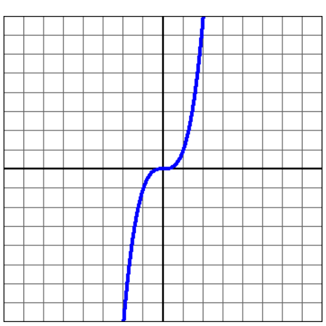
slope of curve is always increasing = first derivative is always increasing

= slope of first derivative is always positive = second derivative is always positive

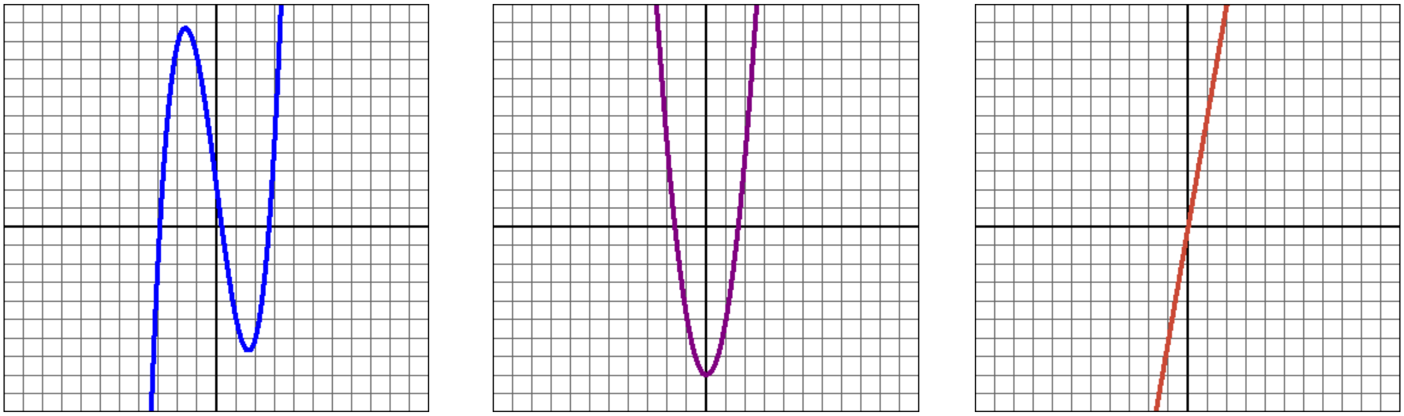
In trying to describe or draw a curve, we'll look for *critical values*, i.e. values at which something significant happens on the curve, i.e. where the first or second derivative (or both) either equals 0 or is undefined.



Consider $y = \sqrt{x} = x^{1/2}$.



For $f = x^3$,



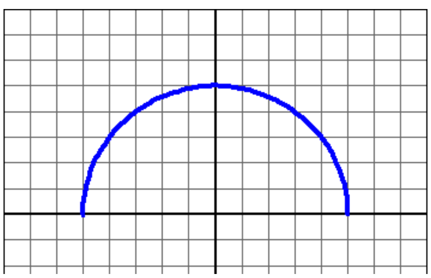
The graphs of $f(x)=x^3-8x+2$, $f'(x)=3x^2-8$, and $f''(x)=6x$ are pictured above. Behaviors of the curve, the first derivative, and the second derivative can be filled into the table below.

interval	f'	f	f''	f
$-\infty < x < -2\sqrt{\frac{2}{3}}$				
$x = -2\sqrt{\frac{2}{3}}$				
$-2\sqrt{\frac{2}{3}} < x < 0$				
$x = 0$				
$0 < x < 2\sqrt{\frac{2}{3}}$				
$x = 2\sqrt{\frac{2}{3}}$				
$2\sqrt{\frac{2}{3}} < x < \infty$				

There are three critical values At $x=-2\sqrt{\frac{2}{3}}$ the curve has a relative maximum: $f'=0$ and $f''<0$. At $x=0$ the curve has a point of inflection: $f''=0$. At $x=2\sqrt{\frac{2}{3}}$ the curve has a relative minimum: $f'=0$ and $f''>0$.

first derivative test:

second derivative test:

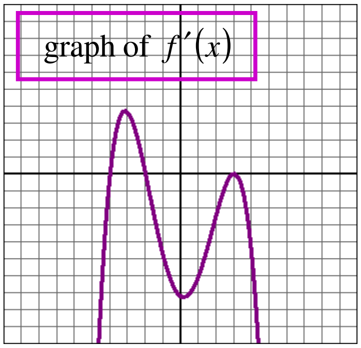


Now consider the function $f(x)=\sqrt{25-x^2}=\left(25-x^2\right)^{1/2}$. Using the general power rule, $f'=\frac{1}{2}\left(25-x^2\right)^{-1/2}*(-2x)=\frac{-x}{\sqrt{25-x^2}}$. When we get the quotient rule in chapter 3, we'll be able to find $f''=\frac{-25}{\left(25-x^2\right)^{3/2}}$.

Note that while f has a domain of $-5 \le x \le 5$, both f' and f'' have a domain of $-5 < x < 5$. Intervals, critical values and interpretations are:

interval	f'	f''	f
$x=-5$			
$-5<x<0$			
$x=0$			
$0<x<5$			
$x=5$			

Now comes the example where we see if we can, without knowing the function itself, describe the behavior of its graph only using information provided by its first and second derivative.



The graph to the left is a graph of $f'(x)$.

Since $f'(x) = 0$ at $x = -4$, $x = -2$, and $x = 3$, we know that f has a maximum, minimum or point of inflection at these values. We can determine which by noting where f' is positive or negative.

interval	$x < -4$	$x = -4$	$-4 < x < -2$
value of f'			

Since the first derivative (slope of f)…

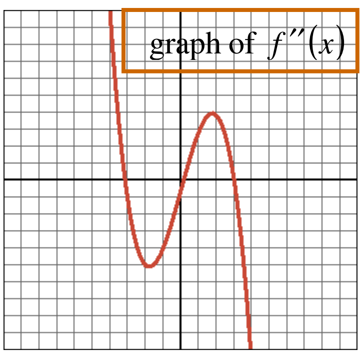
interval	$-4 < x < -2$	$x = -2$	$-2 < x < 3$
value of f'			

Since the first derivative (slope of f) …

interval	$-2 < x < 3$	$x = 3$	$3 < x$
value of f'			

Since the first derivative (slope of f) …

The second derivative below gives us confirmation.



This next graph is a graph of $f''(x)$.

Since $f''(x) = 0$ at $x = -3$, $x \cong 0.1$, and $x = 3$, we'll look in those places for points of inflection. We can determine whether f is concave up or down by determining where f'' is positive or negative.

interval	$x < -3$	$x = -3$	$-3 < x < 0.1$	$x \cong 0.1$	$0.1 < x < 3$	$x = 3$	$3 < x$
value of f''							
f is concave…							

Since the second derivative (indicating concavity)…

Putting all of the information together, we can draw a possible graph for f, which may look something like this: