Calculus 120, section 4.1-4.2 Exponential Functions, Including e^x

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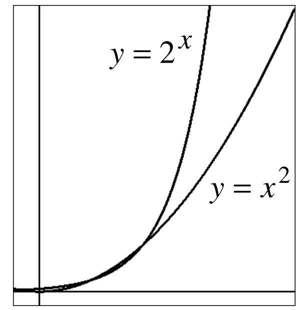
Take a piece of paper, and fold it in half. You've doubled the number of layers—from 1 to 2. Fold it in half again, and you've once again doubled the layers—from 2 to 4. After the next fold you'd have 8 layers. After the next, 16. Then 32. Then 64, 128, 256, 512, 1024, etc. After just 10 foldings, we have the paperback edition of just the first book of *Lord of the Rings* that we're attempting to fold in half.



When x is the base, we have a *power function*. When x is the exponent we have an *exponential function*. The scenario above illustrates the exponential function $y = 2^x$.

If we compare the graph of $y = x^2$ to the graph of $y = 2^x$, we can see that for positive values of x the exponential function grows much more quickly than the power function. (Thus our difficulty in folding a piece of paper in half successive times.) We'll be interested in finding a way to describe the slope of an exponential graph, i.e. a way to find its derivative.

Exponential functions have many applications because they model many kinds of growth and shrinking: e.g. populations, bank deposits, radioactive decay.

Example A: You deposit \$100 into a certificate of deposit which pays 5% each year on the balance current at the time. Find an equation to describe the growth of your money. *Answer*: $A(t) = 100(1.05)^t$

All of the usual properties apply to exponential functions:

$b^x * b^y = b^{x+y}$ $\quad \dfrac{b^x}{b^y} = b^{x-y}$ $\quad \dfrac{1}{b^y} = b^{0-y} = b^{-y}$ $\quad \left(b^x\right)^y = b^{xy}$ $\quad a^x * b^x = (ab)^x$ $\quad \dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$.

Examples B: We can use these properties to simplify expressions and solve equations.

Simplify $\left(2^{-2x} * 2^{x+1}\right)^3$. *Answer*: $2^{-3x+3} = 2^3 * 2^{-3x}$

Simplify $2^{x-1} * 8^{x+3}$. *Answer*: $2^{4x+8} = 2^8 * 2^{4x}$

Simplify $\dfrac{2^{x-1}}{8^{x+3}}$. *Answer*: $2^{-2x-10} = \dfrac{1}{2^{10}} * 2^{-2x}$

Simplify $\left(\sqrt{12}\right)^x * \left(\sqrt{6}\right)^{-x}$. *Answer*: $2^{\frac{1}{2}x}$

We'll use functions with the form $y = Cb^{kx}$ to model all kinds of applications.

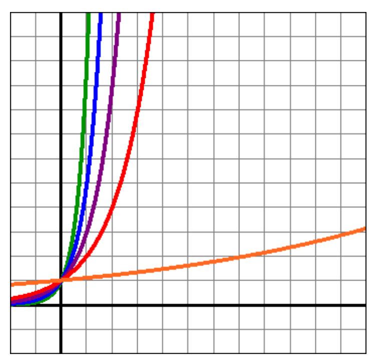
Solve $2^{x^2+3} = 16$. *Answer*: $x = \pm 1$

Solve: $2^x * 3^{x+1} = 108$. *Answer*: $x = 2$

Solve $\left(\sqrt{3}\right)^x = \frac{1}{27}$. *Answer*: $x = -6$

Solve $\frac{5}{1-2^{x+1}} = 10$. *Answer*: $x = -2$

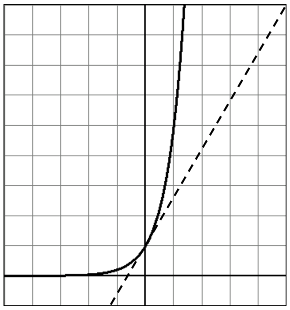
Functions with the basic form $y = b^x$ are actually a family of functions. We'll consider only values for b that are positive. (Negative values of b are extremely problematic, since even and odd values of x would cause y to fluctuate between positive and negative.)



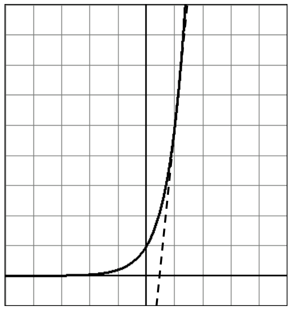
Consider the functions $y = 10^x$, $y = 5^x$, $y = 3^x$, $y = 2^x$, $y = 1.1^x$ pictured in the graph to the right. Note first the similarities: $b^0 = 1$ for all values of $b \neq 0$, so (0, 1) makes a good reference point. Each of the basic exponential functions has a horizontal asymptote $y = 0$. The graphs also have similar shape—the major difference is slope of the curve at specific values of x. Note that at $x = 0$ slope of $y = 10^x$ is steepest; slope of $y = 1.1^x$ is most shallow. Our determination of first derivative will have to reflect this.

The value of the first derivative at a specific point on a curve can be estimated by considering a series of secant lines and using the difference quotient, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

The text does $y = 2^x$; you'll do $y = 3^x$ for homework, so in class we'll do $y = 5^x$ in a process very similar to that of section 1.3.



tangent at $x = 0$



tangent at $x = 1$

First, consider $y = 5^x$ at $x = 0$ where $y = 1$. The slope of the secant line connecting (0, 1) to another point on the curve is given by the formula $\frac{5^{0+h} - 5^0}{h} = \frac{5^h - 1}{h}$. As our second point approaches (0, 1), the slope of the secant line approaches the slope of the tangent line, i.e slope of the tangent = $\lim_{h \to 0} \frac{5^h - 1}{h}$. The table below provides results from successively smaller values of h.

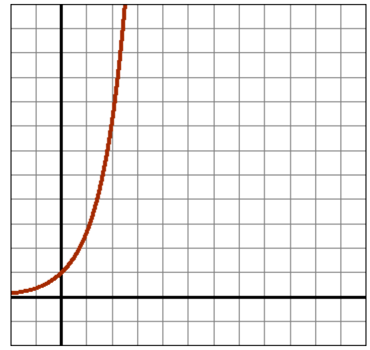
$5^0 =$	$h =$	$5^h \cong$	$5^h - 1 \cong$	$\frac{5^h - 1}{h} \cong$
1	1	5	4	4
1	0.1	1.174618943	0.174618943	1.746189431
1	0.01	1.016224591	0.016224591	1.622459127
1	0.001	1.001610734	0.001610734	1.610733753
1	0.0001	1.000160957	0.000160957	1.609567434
1	0.00001	1.000016095	0.000016095	1.609450864
1	0.000001	1.000001609	0.000001609	1.609439208
1	0.0000001	1.000000161	0.000000161	1.609438043
1	0.00000001	1.000000016	0.000000016	1.609437916

The slope of the tangent to $y = 5^x$ at $x = 0$, and therefore the first derivative of $y = 5^x$ at $x = 0$, is approximately 1.61. (This is an estimate and is not exact. At some later point we'll identify the exact expression of the first derivative of $y = 5^x$ at $x = 0$ as ln (5).)

Next we move to an arbitrary point $(x, 5^x)$ on the graph of $y = 5^x$. The slope of the secant line connecting $(x, 5^x)$ to another point on the curve is given by the formula $\frac{5^{x+h} - 5^x}{h}$, which can be simplified using the properties of exponents: $\frac{5^{x+h} - 5^x}{h} = \frac{5^x * 5^h - 5^x}{h} = \frac{5^x (5^h - 1)}{h}$. As our second point approaches $(x, 5^x)$, the slope of the secant line approaches the slope of the tangent line, i.e slope of the tangent =

$$\lim_{h \to 0} \frac{5^x (5^h - 1)}{h} = \left[\lim_{h \to 0} \frac{(5^h - 1)}{h} \right] * 5^x = \left[\left(5^x\right)' \Big|_{x=0} \right] * 5^x = \left[\frac{d}{dx} \left(5^x\right) \Big|_{x=0} \right] * 5^x \cong 1.61 * 5^x .$$

Recall now a point made above, that in the family of exponential functions, the slope of the tangent at $x = 0$, and therefore the first derivative at $x = 0$, ranges from very shallow to very steep. Somewhere in that family must be a base which has a first derivative equal to exactly 1 at $x = 0$. This number is e, Euler's number. Like π or $\sqrt{2}$, e is an irrational number. The value of e is *approximately* 2.7. (One decimal place will be sufficient for our purposes.) The corresponding function, $y = e^x$, is called the natural exponential function.



Given that for $y = e^x$, $\left.\frac{dy}{dx}\right|_{x=0} = y'(0) = 1$, we can follow a procedure similar to that used above to get a result that is quite interesting. If we consider an arbitrary point (x, e^x) on the graph of $y = e^x$, the slope of the secant line connecting (x, e^x) to another point on the curve is given by the formula $\frac{e^{x+h} - e^x}{h} = \frac{e^x(e^h - 1)}{h}$. As our second point approaches (x, e^x), the slope of the secant line approaches the slope of the tangent line, which means it is also approaching the first derivative, or

$$y' = \frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} = \left[\lim_{h \to 0} \frac{(e^h - 1)}{h}\right] * e^x = y'(0) * e^x = 1 * e^x = e^x.$$

In short, the natural exponential function, $y = e^x$, is special because, unlike other functions, it is its own derivative. This property makes it not only very interesting, but also very useful.

As a side note, this is only one way to define e and approximate its value. There are other definitions, including

$$e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n \text{ and } e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$