Calculus 120, section 4.4 The Natural Logarithm Function

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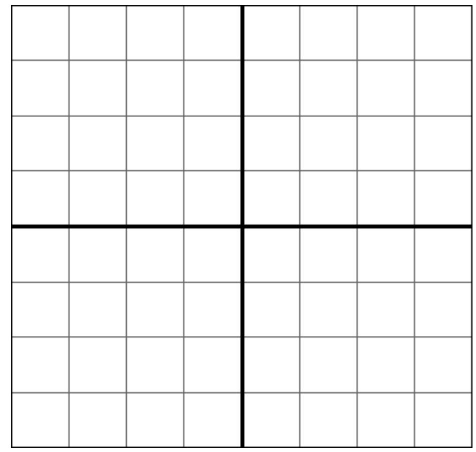
Given a function f and its inverse, f^{-1}, the following will always be true:
1. If $f(a) = b$, then $f^{-1}(b) = a$ (This fact and the statement in point #2 below is actually the same information.)
2. If (a, b) is a point on the graph of f, then (b, a) will be on the graph of f^{-1}.
3. The domain of f = the range of f^{-1}, and the range of f = the domain of f^{-1}.
4. $f \circ f^{-1} = x$ and $f^{-1} \circ f = x$. To show that two functions are inverses you must do *both* compositions.
5. The graph of f and the graph of f^{-1} are symmetric with respect to the line $y = x$.



A *logarithm* function is a constructed inverse for an exponential function. The *natural logarithm* function, $y = \ln(x)$, is the inverse of the natural exponential function, $f(x) = e^x$. Applying the above:
1. For one example, $f(0) = e^0 = 1$ and $f^{-1}(1) = \ln(1) = 0$.
2. For one example, (0, 1) is on the graph of $f(x) = e^x$ and (1, 0) is on the graph of $f^{-1}(x) = \ln(x)$.
3. The domain of $f(x) = e^x$ is the range of $f^{-1}(x) = \ln(x)$: $-\infty < x < \infty$.
   The range of $f(x) = e^x$ is the domain of $f^{-1}(x) = \ln(x)$: $0 < x < \infty$.
   Note also that while the graph of $f(x) = e^x$ has a horizontal asymptote at $y = 0$, the graph of $f^{-1}(x) = \ln(x)$ has a vertical asymptote at $x = 0$.
4. $f \circ f^{-1} = e^{\ln x} = x$ and $f^{-1} \circ f = \ln\left(e^x\right) = x$.
5. The graphs of $f(x) = e^x$ and $f^{-1}(x) = \ln(x)$ are symmetric with respect to the line $y = x$.

In Lecture 4.2 we determined that the slope of $f(x) = e^x$ is 1 at the point (0, 1), i.e. $f'(0) = 1$.
By symmetry, the slope of $f^{-1}(x) = \ln(x)$ is 1 at the point (1, 0), i.e. $\left(f^{-1}\right)'(1) = 1$.
Also note that while both functions are increasing over their entire domain, $f(x) = e^x$ is concave up, $f^{-1}(x) = \ln(x)$ is concave down.
In Lecture 4.5, we'll use points 1, 2, and 5 from above to determine the derivative of $f^{-1}(x) = \ln(x)$.

Example A: Simplify $\ln\left(\sqrt[5]{e^3}\right)$. *Answer*: $\frac{3}{5}$

Example B: Simplify $e^{\ln(x+2)}$. *Answer*: $x + 2$

Example C: Simplify $e^{\ln(x)+2}$. *Answer*: xe^2

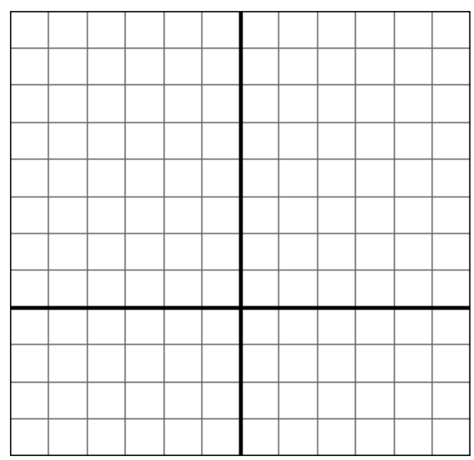
Example D: Solve $5\ln(x-1)+4=0$. *Answer*: $e^{-4/5}+1$

Example E: Solve $\ln(x^2-1)=4$. *Answer*: $\pm\sqrt{e^4+1}$

Example F: Given $f(x)=5^x$, find $f'(x)$. *Answer*: $(\ln 5)*5^x$

Example G: Find relative extrema of $f(x)=5x+e^{-2x}$.

Find first derivative and set it equal to 0:



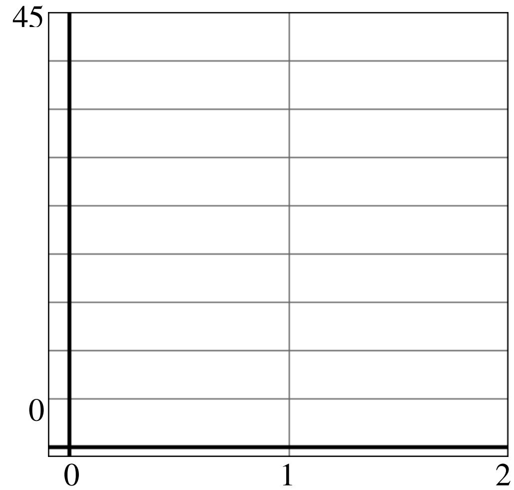
Use f'' to determine concavity:

We conclude that the point $\left(-\frac{1}{2}\ln\frac{5}{2},\ \frac{5}{2}-\frac{5}{2}\ln\frac{5}{2}\right)$ is an absolute minimum.

slant asymptote:

Example H: The concentration of a drug in the bloodstream t hours after injection into a muscle is given by $c(t) = 9\left(e^{-0.3t} - e^{-3t}\right)$ units. Find the time at which the concentration of the drug in the bloodstream is at its maximum.

Example I (logistic growth curve—more about these in chapter 5): Fish population as a function of time is given by $P(t) = \dfrac{80}{2 + 3e^{-10t}}$. Sketch the graph.



The grid above is in a window [-0.1, 2,1] by [-1, 45,5].

domain:

P-intercept (vertical axis):

t-intercept (horizontal axis):

maxima/minima?

point of inflection?

To the left of the point of inflection the graph is concave up; to the right it is concave down.

asymptotes?

Note how this logistic growth curve is very steep then seems to level off quickly.