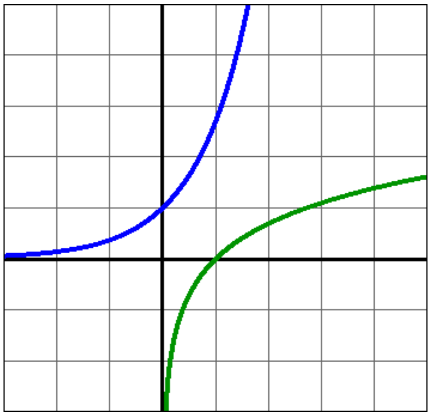
Calculus 120, section 4.5 The Derivative of ln(*x*)

notes by Tim Pilachowski



The *natural logarithm* function, $y = \ln(x)$, is the inverse of the natural exponential function, $y = e^x$. In Lecture 4.2 we determined that the slope of $y = e^x$ is 1 at the point (0, 1), i.e. $\frac{d}{dx}\left(e^x\right)\Big|_{x=0} = 1$. By symmetry, the slope of $y = \ln(x)$ should also be 1 at the point (1, 0), i.e. $\frac{d}{dx}(\ln x)\Big|_{x=1} = 1$. Also recall that $y = \ln(x)$ is increasing and the graph is concave down over its entire domain The formula we use for the derivative of ln(*x*) must meet all of these conditions.

Finding a derivative formula for ln(*x*) is actually quite simple. First note that since $e^{\ln x} = x$, then $\frac{d}{dx}\left(e^{\ln x}\right) = \frac{d}{dx}(x) = 1$. By the chain rule, $\frac{d}{dx}\left(e^{\ln x}\right) = e^{\ln x} * \frac{d}{dx}(\ln x) = x * \frac{d}{dx}(\ln x) = 1 \rightarrow \frac{d}{dx}(\ln x) = \frac{1}{x}$.

Note that $\frac{d}{dx}(\ln x)\Big|_{x=1} = \frac{1}{x}\Big|_{x=1} = 1$. Also $\frac{d}{dx}(\ln x) = \frac{1}{x} > 0$ and $\frac{d^2}{dx^2}(\ln x) = -\frac{1}{x^2} < 0$ for all *x* in the domain of ln(*x*). In other words, all of the necessary conditions listed above have been met. The examples below will utilize this formula along with the product rule, quotient rule and chain rule.

Example A: Given $h(x) = x^3 * \ln x$ find the first and second derivative. *Answers*: $x^2(1 + 3\ln x)$; $x(5 + 6\ln x)$

Example B: Given $f(x) = \frac{x^3}{\ln x}$ find the first derivative. *Answer*: $\frac{x^2(3\ln x - 1)}{[\ln x]^2}$

Example C: Given $g(x) = \frac{\ln x}{x^3}$ find the first derivative. *Answer*: $\frac{1 - 3\ln x}{x^4}$

Carefully note the placement of coefficients when finding derivatives.

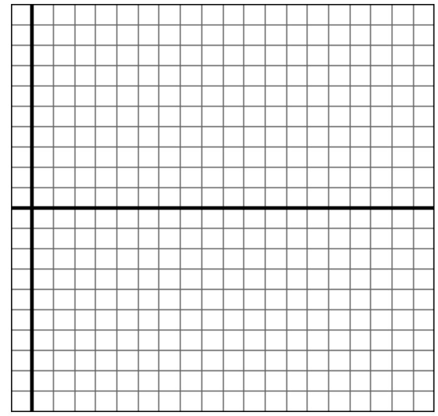
constant multiple rule	chain rule
$m(x) = k\ln(x)$	$n(x) = \ln(kx)$

Example D: Give $h(x) = \ln(x^3)$, find the first and second derivatives. *Answers*: $\frac{3}{x}$; $-\frac{3}{x^2}$

When using the chain rule, it is extremely important to correctly identify the "outside" and "inside" functions. Check that your composition is set up correctly.

Example E: Give $h(x) = [\ln x]^5$, find the first and second derivatives. *Answers*: $\frac{5(\ln x)^4}{x}$; $\frac{20(\ln x)^3 - 5(\ln x)^4}{x^2}$

We can use the above to sketch the graph of $h(x) = [\ln x]^5$.



asymptotes:

y-intercept:

x-intercepts:

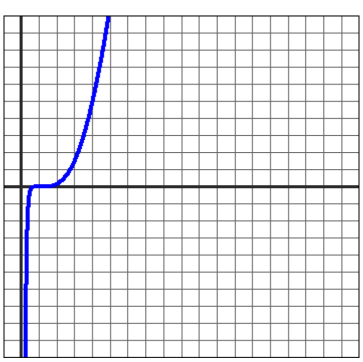
possible extrema:

possible points of inflection:

A table of signs tells us concavity:

interval			
sign of h''			
concavity			

The point of inflection at $(e^4, 1024) \approx (54.598, 1024)$ is far outside the standard window on a graphing calculator. One could zoom out a good bit, or use different scales for x- and y-axes—but then the other characteristics of the graph would be obscured. Note the three versions of the graph pictured below.



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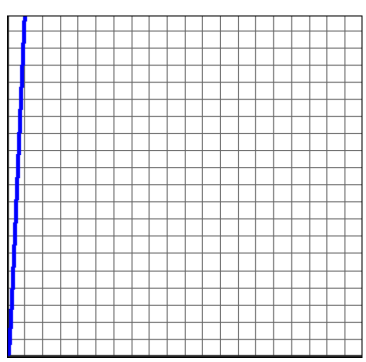

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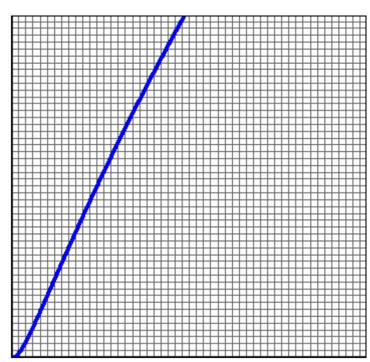

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