Calculus 120, section 4.6 Properties of ln(*x*)

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Recall the properties of exponential functions:

$$b^x * b^y = b^{x+y} \qquad \frac{b^x}{b^y} = b^{x-y} \qquad \frac{1}{b^y} = b^{0-y} = b^{-y} \qquad \left(b^x\right)^y = b^{xy} \qquad a^x * b^x = (ab)^x \qquad \frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x.$$

Logarithm functions, and in particular the *natural logarithm* function, $y = \ln x$, have properties of their own, related to but not exactly the same as the exponential function properties. While the logarithm properties apply to logarithms in any base, since we will be rewriting all bases in terms of the natural logarithm, our focus will be entirely on the logarithm properties applied to ln(*x*). Of the four properties listed in the text, two are vital. The other two can be easily derived from these.

$$\ln(xy) = \ln x + \ln y \qquad \ln\left(x^a\right) = a \ln x$$

To prove the first: Note that $e^{\ln(xy)} = xy$. But since $x = e^{\ln x}$ and $y = e^{\ln y}$, we can use the properties of exponents to write $e^{\ln(xy)} = x * y = e^{\ln x} * e^{\ln y} = e^{\ln x + \ln y}$. Since the exponents must be equal, we're done.

Proof of the second is along similar lines: $e^{\ln\left(x^a\right)} = x^a = \left(e^{\ln x}\right)^a = e^{a * \ln x}$.

Knowing that $\frac{1}{x} = x^{-1}$, the other two properties given by the text become fairly obvious:

$$\ln\frac{1}{x} = \ln\left(x^{-1}\right) = -\ln x \qquad \ln\frac{x}{y} = \ln\left(xy^{-1}\right) = \ln x + \ln\left(y^{-1}\right) = \ln x - \ln y.$$

Example A: Simplify ln 60 – ln 4 – ln 5. *Answer*: ln(3)

Example B: Simplify $e^{3\ln x + 2\ln y}$. *Answer*: x^3y^2 , $x > 0$ and $y > 0$

Restrictions on the original expression restrict the domains of the simplified version: $x > 0$, $y > 0$.

Example C: Using properties of logarithms, expand and simplify $\ln\left(x^2 e^x\right)$. *Answer*: $2\ln(|x|) + x$

Absolute value is necessary to retain the domain of the original: $-\infty < x < 0$ and $0 < x < \infty$.

Example D: Using properties of logarithms, expand $\ln\left[\dfrac{x+1}{(x^2-5)(3x+7)^5}\right]$.

Answer: $\ln(x+1)-\ln(x^2-5)-5\ln(3x+7)$

Be careful when using the logarithm properties! They *do not* allow us to expand this example any more—the properties apply when a product or quotient is inside the logarithm, but *not* when there is a sum or difference.

Recall from Lecture 4.5: $\dfrac{d}{dx}(\ln x)=\dfrac{1}{x}$. Combining this derivative with the properties above gives us a means of finding the derivative of some complicated-looking functions.

Example C revisited: Given $f(x)=\ln(x^2e^x)$, find the first and second derivatives. *Answers*: $\frac{2}{x}+1$; $-\frac{2}{x^2}$

Note that the domain, $-\infty < x < 0$ and $0 < x < \infty$, is retained by the denominators of x in f' and x^2 in f''.

Example D revisited: Given $m(x)=\ln\left[\dfrac{x+1}{(x^2-5)(3x+7)^5}\right]$, find the first derivative. *Answer*: $\frac{1}{x+1}-\frac{2x}{x^2-5}-\frac{15}{3x+7}$

While we *could* have used the chain rule and the quotient rule, using the logarithm properties made the process easier. Note that in the process of differentiating ln(something), we still needed to make use of the chain rule.

The natural logarithm function, ln(x), can be used in a process called *logarithmic differentiation* to ease the differentiation of products and quotients involving multiple terms. Note that for any function $f(x)=\ln[g(x)]$, by the chain rule $f'(x)=\frac{1}{g(x)}*g'(x)=\frac{g'(x)}{g(x)}$.

Example E: Given the polynomial $g(x)=(x+3)(x+1)^2(x-1)^3$, find the first derivative.
As Example D in Lecture 3.1, we used the chain rule and successive applications of the product rule to show $g'=3(x+3)(x-1)^2(x+1)^2+2(x+3)(x+1)(x-1)^3+(x+1)^2(x-1)^3$.

Using logarithmic differentiation,

(a) Take the natural logarithm of both sides and use logarithm properties to expand:

$$\ln[g(x)]=\ln\left[(x+3)(x+1)^2(x-1)^3\right]=\ln(x+3)+2\ln(x+1)+3\ln(x-1)$$

(b) Take the derivative of ln [$g(x)$]:

$$\frac{g'(x)}{g(x)}=\frac{1}{x+3}+\frac{2}{x+1}+\frac{3}{x-1}$$

(c) Solve algebraically for $g'(x)$:

$$g'(x)=\left[\frac{1}{x+3}+\frac{2}{x+1}+\frac{3}{x-1}\right]*g(x)$$

(d) Back-substitute for $g(x)$:

$$g'(x)=\left[\frac{1}{x+3}+\frac{2}{x+1}+\frac{3}{x-1}\right]*\left[(x+3)(x+1)^2(x-1)^3\right]$$

A little multiplication and canceling shows that this is equal to the derivative given above.

Example F: Find the first derivative of $h(x)=\frac{(x^3-2)(x^2-3)^4}{\sqrt{8x-5}}$.

Answer: $\left[\frac{3x^2}{x^3-2}+\frac{8x}{x^2-3}-\frac{4}{8x-5}\right]*\frac{(x^3-2)(x^2-3)^4}{\sqrt{8x-5}}$