Calculus 120, section 5.1 Exponential Growth & Decay

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In Lecture 4.3, we noted that a differential equation of the form $y' = ky$ must have an exponential base *e* as its solution, i.e. $y = Ce^{kx}$ for some coefficient *C*. If we take this basic form, and define *x* as representing time, then it is a simple process to note that when time = *x* = 0, $y = Ce^{k(0)} = C$. In other words, *C* is an initial value, the amount we start out with at time = 0. In this section we'll use two versions of this basic exponential form. (In a later section we'll add a third.)

1) For population growth, we have $P(t) = P_0 e^{kt}$, where $P(t)$ is the population after time = *t*, P_0 is the initial population, *e* is Euler's number, and *k* is called the *growth constant*.

2) For decay (radioactive and otherwise), we have $P(t) = P_0 e^{-\lambda t}$, where $P(t)$ is the amount remaining after time = *t*, P_0 is the initial amount, *e* is Euler's number, and λ is called the *decay constant*.

Notice that the only real difference is that the coefficient of *t* is positive for exponential growth and negative for exponential decay.

Example A: A country's population grows according to the model $P(t) = 72e^{0.025t}$ where $t = 0$ represents the year 1980 and P = population in millions. What was the population in 1990 (rounded to the nearest 10,000)?
Answer: 92,450,000 people

Example B: Yeast in a culture increases from 4 grams to 10 grams after 7 hours. Find the growth constant *k*.
Answer: $\frac{\ln(2.5)}{7}$

Note that $k = \frac{\ln(2.5)}{7}$ is the *exact* answer; 0.13 is a *decimal approximation*. Translated into words, it means that the yeast culture is growing at a rate of about 13% per hour.

Example B extended: How long will it take for the original yeast culture to triple in size? Give both an exact answer and an approximation to the nearest hundredth of an hour.

The exact answer is $\frac{7\ln(3)}{\ln(2.5)}$ hours. (Note that there is no logarithm property that will allow us to combine the quotient of two logarithms into a simpler form!) The approximate answer is (8.392… rounded to) 8.39 hours.

Example C: In 1985, the membership in the Red Dwarf Party (an ultra-moderate group advocating curry for dinner five days a week) was 3.60 million. By 1990 it had grown to 5.76 million. Express membership as a natural exponential function of time. *Answer*: $P(t) = 3.6e^{\frac{\ln(1.6)}{5}t}$

The practice of carbon-14 dating is based in the scientifically-supported premise that the proportion of radioactive carbon ^{14}C to non-radioactive carbon ^{12}C present in the atmosphere and in most living tissue has remained constant over tens of thousands of years. As long as something is alive, the amount of ^{14}C is maintained. When it dies the amount of ^{12}C remains constant but the amount of ^{14}C lessens through radioactive decay. Measuring the proportion of ^{14}C to ^{12}C that remains in an organism allows for a rough estimate of the length of time an organism has been dead.

Example D: ^{14}C has a decay constant (λ) of 0.00012. If a fossil is found that has 75% of the ^{14}C level in the atmosphere, estimate (to the nearest year) how old it is. *Answer*: 2397 years

Example D extended: The *half-life* of a radioactive isotope is the amount of time it takes for an initial amount to decay to half of that initial amount. Solving for a half-life is solving for a time, *t*. Find the half-life of ^{14}C.
Answer: $\frac{\ln(0.5)}{-0.00012} \cong 5776$ years

Example E: The half-life of krypton-92 (^{92}Kr) is 3 seconds. If you begin with 100 g, how much is left after 3 seconds? 6 seconds? 9 seconds? 12 seconds?
Using the definition of half-life, each three seconds that passes reduces the amount to half of what it was.
After 3 seconds you'd have half of 100 = 50 g.
After 6 seconds (3 more) you'd have half of 50 = 25 g.
After 9 seconds (3 more) you'd have half of 25 = 12.5 g.
After 12 seconds (3 more) you'd have half of 12.5 = 6.25 g.

Example E extended: After how much time would you have 2 g of the original 100 g? Give both an exact answer and an approximation to the nearest thousandth of a second. *Answers*: $\frac{3\ln(0.02)}{\ln(0.5)} \cong$ 16.932 seconds