Calculus 120, section 5.2 Compound Interest

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Example A (repeated from Lecture 4.1): You deposit \$100 into a certificate of deposit which pays 5% each year on the balance current at the time. Find an equation to describe the growth of your money.
Define $A(t)$ = amount of money accumulated after t years. The table below summarizes the growth over 5 years.

t	interest earned	A = accumulated amount
	percent * [current balance]	{current balance} + {percent * [current balance] } = [current balance] * (1 + percent)
0	0	{100}
1	0.05 * [100]	{100} +{ 0.05[100] } = 100(1 + 0.05)
2	0.05 * [100(1 + 0.05)]	$\{100(1+0.05)\} + \{0.05[100(1+0.05)]\}$ = $[100(1+0.05)] * (1 + 0.05)$ = $100(1+0.05)^2$
3	0.05 * $[100(1+0.05)^2]$	$\{100(1+0.05)^2\} + \{0.05[100(1+0.05)^2]\}$ = $[100(1+0.05)^2] * (1 + 0.05)$ = $100(1+0.05)^3$
4	0.05 * $[100(1+0.05)^3]$	$\{100(1+0.05)^3\} + \{0.05[100(1+0.05)^3]\}$ = $[100(1+0.05)^3] * (1 + 0.05)$ = $100(1+0.05)^4$
5	0.05 * $[100(1+0.05)^4]$	$\{100(1+0.05)^4\} + \{0.05[100(1+0.05)^4]\}$ = $[100(1+0.05)^4] * (1 + 0.05)$ = $100(1+0.05)^5$

We can use the pattern to state a general formula for interest added annually for n years:

n	0.05 * $[100(1+0.05)^{n-1}]$	$\{100(1+0.05)^{n-1}\} + \{0.05[100(1+0.05)^{n-1}]\}$ = $[100(1+0.05)^{n-1}] * (1 + 0.05)$ = $100(1+0.05)^n$

So, If we left our money there for 10 years (t = 10) we'd have $A(10) = 100(1+0.05)^{10} \cong \$162.89$.
If the interest was compounded quarterly, the 5% annual rate would be divided up among the four quarters, and the number of interest calculations would be n = 4(10): $A(10) = 100\left(1+\frac{0.05}{4}\right)^{4(10)} \cong \$164.36$.

For interest compounded monthly, we'd have: $A(10) = 100\left(1+\frac{0.05}{12}\right)^{12(10)} \cong \$164.70$.

For interest compounded daily, we'd have: $A(10) = 100\left(1+\frac{0.05}{365}\right)^{365(10)} \cong \$164.87$.

For different principals, P, rates of interest, r, compounding periods, m, and numbers of years, t, we can generalize: $A(P, r, m, t) = P\left(1+\frac{r}{m}\right)^{mt}$.

Using this formula we could recalculate our balance compounding every hour, second, or fraction of a second.
What happens if we increase the number of times interest is calculated and approach infinity?

At the end of Lecture 4.2, we noted that an "alternate" way of defining e is $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

If we now take our generalized formula, and replace $\frac{r}{m} = \frac{1}{n} \Rightarrow nr = m$, and take $\lim_{n\to\infty}$ we have

$$A(P,r,t)=\lim_{n\to\infty} P\left(1+\frac{1}{n}\right)^{nrt}=P\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{rt}=Pe^{rt}.$$

This is called "continuous compounding".

Example A again: For an initial balance of \$100 and an annual interest rate of 5% compounded continuously over 10 years, calculate the closing balance rounded to the nearest penny. *Answer*: \$164.87

If you are calculating interest on a hand calculator, the continuous compounding formula is much easier to use. If you are working at a bank or investment firm, doing a massive number of this type of calculation, the continuous compounding formula uses a lot less computer time and memory.

Example B: Estimate the amount of time (rounded to the nearest tenth of a year) it would take an investment of \$100 to double in value at an interest rate of 5% annually with continuous compounding.
Answer: exact 20 ln(2) years; approximate 13.9 years

Another advantage of the continuous compounding formula is that it is another form of the general exponential growth model $P(t)=P_0e^{kt}$, where $P(t)$ is the value of an investment after time = t, P_0 is the initial value, e is Euler's number, and k is the annual rate of growth. Note that, like any exponential growth equation, $A=Pe^{rt}$, solves the differential equation $y'=ky$: in this case $A'=rA$ and r is our rate of growth.

Example C: One business application is estimating the rate of growth of an investment for purposes of computing capital gains taxes or comparing the return from two investments of different amounts over different periods of time.

The value of some utility stocks has grown from \$300 to \$450 over the past 5 years. This corresponds to what rate of interest if the money had been put into a certificate of deposit with continuous compounding?
Answer: exact $\frac{1}{5}\ln\left(\frac{3}{2}\right) = 20\ln\left(\frac{3}{2}\right)\%$; approximate 8.1%

Example D: The next application example for this section involves *present value*: the value an investment needs today (i.e. in the present time) to end with a specified value at some future time.

You expect to retire in 40 years and would like to have \$100000 in the bank. With an expected interest rate of 5% and continuous compounding, how much should you put into a certificate of deposit now?
Answer: \$13533.53

The continuous compounding formula, $A(P, r, t) = Pe^{rt}$, involves four variables:
A = accumulated amount (the amount of money you end up with)
P = principal invested (the amount of money you started with)
r = annual rate of interest
t = time in years
A homework or test question could ask for any one or a combination of these.

Example E: An initial investment of \$500 has grown to \$674.93 over the past 5 years. What is its expected value in another 5 years? *Answer*: exact $500e^{2\ln\left(\frac{674.93}{500}\right)}$; to the nearest penny \$911.06