Calculus 120, section 5.4 Further Exponential Models

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So far in Chapter 5, we've looked at variations of the same model of exponential growth or decay,

$$f(x) = Ce^{kx} \qquad P(t) = P_0 e^{kt} \qquad P(t) = P_0 e^{-\lambda t} \qquad A = Pe^{rt}$$

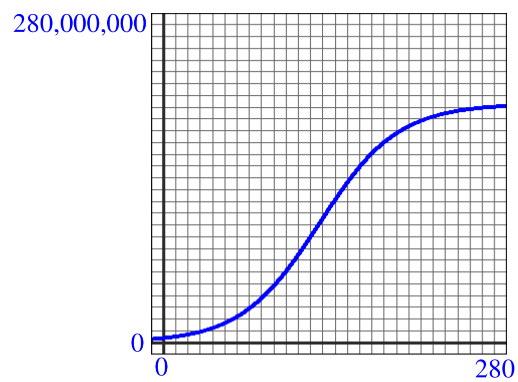
with the only real difference being the letter used for variables in different applications. In each case, the derivative has been of the form $y' =$ (coefficient from the exponent) times y, leading to the conclusion that the (coefficient from the exponent) is the rate of growth or decay. [Note: This type are formulae you should know.]

But this is not the only kind of formula involving the natural exponential function $y = e^x$. You won't be required to memorize the formulae in the following examples. You also do not need to know the methods of deriving the formulae as done in the text. You *do* need to know how to work with them when given on the test.

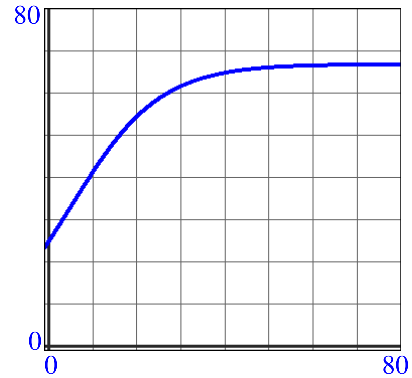
Example A: The exponential growth model $P(t) = P_0 e^{kt}$ applied to populations of people or animals has a serious flaw: In the real world the number that can survive is limited by the amount of space and the number of resources available. A *logistic growth curve* is more appropriate for long-term applications.

A Belgian sociologist, Verhulst, developed a model used by the U.S. Census Bureau for a number of years: $P(t) = \dfrac{202.31}{1 + e^{3.98 - 0.314t}}$ where P = population in millions and $t = 0$ represents the year 1790, $t = 1$ represents 1800, etc. a) What was the population of the U.S. in 1790 (rounded to the nearest 100,000)? b) What population did the model predict for 1940? c) What is the theoretical "upper limit" according to this model?

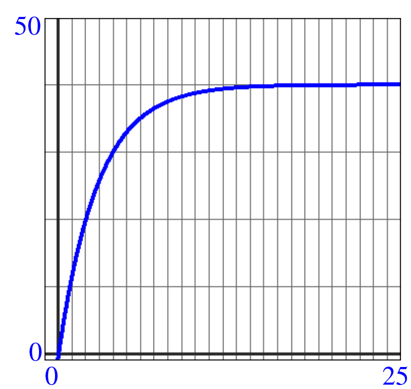
Answers: 3,700,000 people; 136,500,000 people; 202,310,000 people



Example B: The population of deer in a wildlife preserve is modeled by $P(t) = \dfrac{200}{3 + 5e^{-0.1t}}$. a) What was the number of deer at the beginning? b) What is the theoretical "upper limit" according to this model? c) How quickly is the population growing after 10 years? *Answers*: 25 deer; 66-67 deer; $\dfrac{100}{e\left(3+{}^{5}\!/\!_{e}\right)^{2}} \approx 1.6$ deer per year



Example C: The number of units a new worker can produce on an assembly line after *t* days on the job is given by the formula $N(t) = 40 - 40e^{-0.35t}$. This function is called a *learning curve*. a) How many units can the worker make when she or he first begins? b) What is the maximum number he or she can be expected to make? c) Estimate how long it will take for the worker to be able to produce 30 units. d) What is the worker's rate of production? *Answers*: 0 units; 40 units; $\dfrac{20\ln(4)}{7} \approx 4$ days; $14e^{-0.35t}$ units per day



Example D: In an educational psychology study a researcher determines that the number of new historical dates a person can remember after *t* days is given by the formula $H(t) = 125\left(\frac{1+e}{1+e^t}\right), \quad t \geq 1$. This function is called a *forgetting curve*. a) How many dates can a person remember on day 1? b) Estimate how long it will take for a person to remember 25 dates. c) At what rate is the person forgetting? ***Note***: (1 + *e*) is a number, and is treated in the same way as the constant coefficient 125.

Answers: 125 dates; ln(4 + 5*e*) ≈ 3 days; $\frac{-125(1+e)*e^t}{(1+e^t)^2}$ dates per day