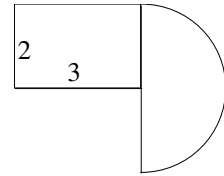


Calculus 120, section 6.4 Areas in the x - y Plane

notes by Tim Pilachowski

Example A: Find the area of the geometric figure pictured to the right. *Answer:* $6 + 2\pi$



In Lectures 6.2 and 6.3 we were able to equate the area under a curve on an interval $a \leq x \leq b$ with the definite integral from a to b : $\int_a^b f(x) dx$. Combining this concept with geometric ideas such as those used in Example

A supports the assertion of some of the properties of integrals already used in Lecture 6.1.

$$\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

$$\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

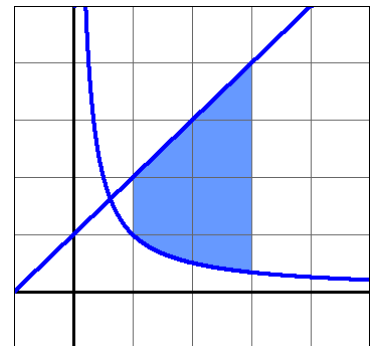
$$\int_a^b k * f(x) dx = k * \int_a^b f(x) dx$$

We'll apply these properties to some new scenarios.

Example B: Find the area between the curves $f(x) = x + 1$ and $g(x) = \frac{1}{x}$ on the interval $1 \leq x \leq 3$.

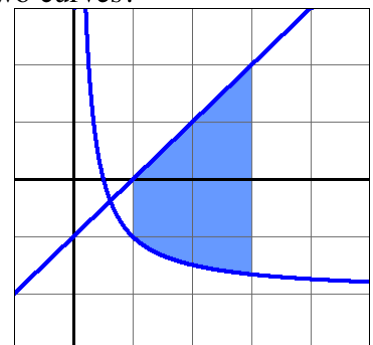
As long as $f(x) > g(x)$ for all x in the interval, we can use the second property above and subtract the areas.

Answer: $6 - \ln 3$



Example C: Find the area between the curves $f(x) = x - 1$ and $g(x) = \frac{1}{x} - 2$ on the interval $1 \leq x \leq 3$.

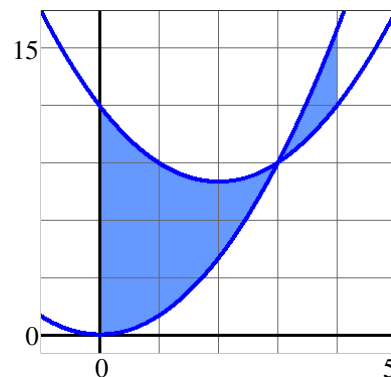
The shape remains the same because the curves from Example B were simply shifted down by 2. The area should be the same. However, in this case some of the area lies below the x -axis, and would thus be a negative area (as noted in Lecture 6.3). Does this affect the calculation of area between the two curves?



So the integral calculation gives us the same result as in Example B: $\text{area} = 6 - \ln 3$. It doesn't matter whether the two curves lie above or below the x -axis, only that we subtract the "higher" curve minus the "lower".

Example D: Find the area between the curves $y = x^2$ and $y = x^2 - 4x + 12$ on the interval $0 \leq x \leq 4$.

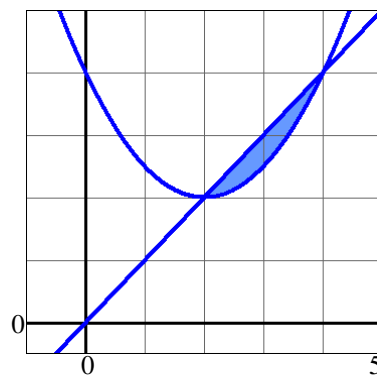
Looking at the graph (window $[-1, 5]$ by $[-1, 17]$), $y = x^2 - 4x + 12$ lies above $y = x^2$ for a portion of the interval, but is below for the rest of the interval. We need to determine where the two intersect, then set up two integrals: one for each portion. *Answer: 20*



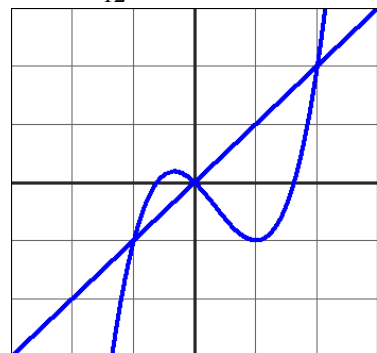
Example E: Find the area between the curves $y = 2x$ and $y = x^2 - 4x + 8$.

In this case, although no interval is specified, by looking at the graph (window $[-1, 5]$ by $[-1, 10]$) we can see that there are two points of intersection. Between these two points the area is bounded, and lies between the two curves. We need to solve for the intersections to find the boundaries (or limits) of integration. *Answer: $\frac{4}{3}$*

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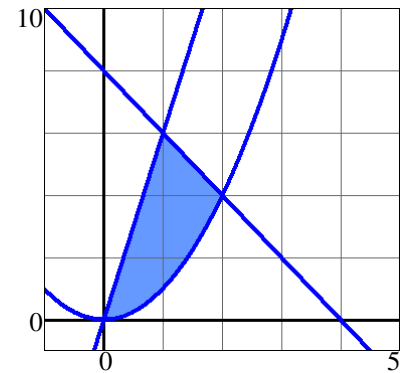
Example F: Find the area between the curves $f(x) = x^3 - x^2 - x$ and $g(x) = x$. *answer: $\frac{37}{12}$*



Example G: Find the area bounded by $y = x^2$, $y = 6x$ and $y = 8 - 2x$ on the interval $0 \leq x \leq 2$.

Looking at the graph (window $[-1, 5]$ by $[-1, 10]$), one of the corners of the bounded area is clearly $(0, 0)$.

We'll need to solve for the others. *Answer:* $\frac{16}{3}$



The key to all of these is looking at the graph to determine which function is above the other, finding intersections as needed, then setting up and evaluating the appropriate integrals.

In the next section we'll delve into applications which use all of the integration concepts developed so far. If a function represents a *rate of change*, we'll use an integral (= area under the curve) to determine an *amount*. For example, from a velocity function, which is the rate of change of position with respect to time, we can integrate to determine the distance traveled. Another example: Knowing how costs are changing with respect to the number of units produced (Marginal Cost) we can integrate to find the amount of Cost.