Calculus 120, section 6.5 Applications of the Definite Integral

notes by Tim Pilachowski

So far in this class, we have spent chapters 1 through 5 beginning with a function that represents an *amount*. The derivative (= slope of the curve) gives us a *rate of change*.

In chapter 6, we are beginning with a *rate of change* and use the integral (= area under the curve) to determine an *amount*.

In applications, it is first important to determine the question—to identify what sort of answer you are looking for: Is it an *amount* or is it a *rate of change*? Next identify the nature of the information you have: Is it an *amount* or is it a *rate of change*?

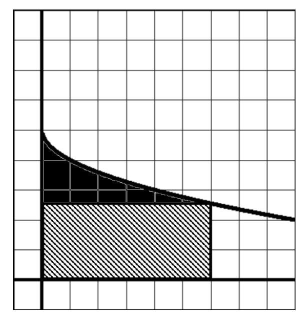
Example A: For a retail company, the monthly marginal cost when x units have been purchased is $50 - 0.08x$ dollars. Their monthly fixed costs are \$700. a) Determine the cost for producing x units. b) What is the cost of raising monthly purchases from 10 to 15 units?
For part a) we want to know a cost, the *amount* of money needed to purchase items. The cost to purchase x units is the Cost function $C(x)$. The information give is a *rate of change*—the marginal cost is $C'(x)$, the derivative of the Cost function. "Monthly fixed costs are \$700" is an *amount* when $x = 0$, i.e. $C(0) = 700$. We can find $C(x)$ from $C'(x)$ using the integral. *Answer*: $C(x) = -0.04x^2 + 50x + 700$

For part b) we want to know a cost, an *amount*. The information involves a *rate of change*: raising the number of units from 10 to 15. The numbers 10 and 15 set definite boundaries for us. We need a definite integral.
Answer: \$245

Example B: An object falls from a high-altitude balloon with a speed of $32t$ feet per second. How far does the object fall during the first 4 seconds?
"How far" asks for a distance, which is an *amount*. The velocity is a *rate of change* of distance with respect to time: $v(t)$ in $\frac{\text{feet}}{\text{second}}$ is $\frac{\text{change in distance}}{\text{change in time}}$. "The first four seconds" gives us boundaries of $t = 0$ and $t = 4$.
Answer: downward 256 feet

Example C: The rate at which gasoline was consumed in the U.S. was approximately $r(t) = 0.075t + 1.7$ billion gallons per year from 1964 ($t = 0$) to 1976. Find the number of gallons of gas consumed from 1964 to 1976. The equation give is a *rate of change* of consumption. The “number of gallons” asked for is an *amount*. “From 1964 to 1976” gives us boundaries: $t = 0$ to $t = 12$. We need a definite integral. *Answer*: 25.8 billion gallons

Another application using integrals comes from the business world. The text does a *consumer surplus* problem as Example 4. The explanation below is designed to add to your understanding of why the process works the way it does. Let p = price and x = the number of units sold. A demand curve illustrates the reality that as price of goods increases, fewer people will buy, and the demand goes down. The function $p = f(x)$ will have a negative slope. A predictable number of items $x = A$ will be sold if the price is set at a particular level $B = f(A)$. The area of the rectangle in the picture has area $A * B$ = number sold times price = revenue generated by sales. Note that the top edge of the rectangle is the horizontal line function $p = f(A) = B$. The area under the curve above the rectangle represents the money saved by the people who were willing to pay a higher price: the *consumer’s surplus*. We already know that area under a curve can be found by integrating. Thus



consumer’s surplus = area under demand curve – area of rectangle

= [area under $p = f(x)$] – [area under $p = f(A) = B$]

$$= \int_0^A \left[f(x) - B\right] dx$$

Example D: Let $p = 15 - 0.5x$ represent a demand function where p is price in dollars and x is the number of units sold. Let the level of demand = 7. a) What was the price? b) How much revenue was generated? c) What was the consumer’s surplus? *Answers*: \$11.50; \$80.50; \$12.25

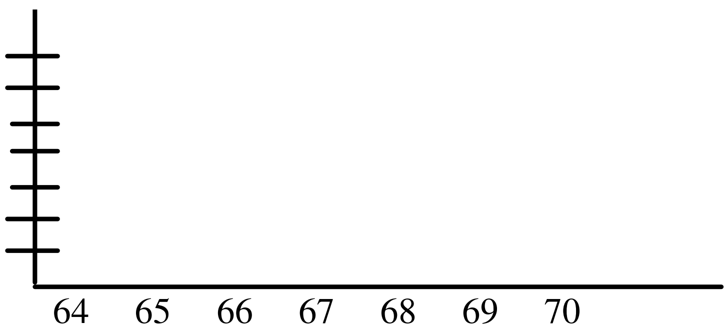
You’ll need Example 5 and the formula at the top of the page following it in the text to do some of the assigned homework. It is similar to the continuous compounding done in section 5.2, but instead of one deposit there are a series of deposits (a *continuous income stream*) that increase the balance along with the interest. If you think of each new deposit as an $A = Pe^{rt}$ calculation where A = accumulated balance, the sum of balances from each deposit forms a Riemann sum, i.e. an integral calculation.

For our final applications of integrals, we'll take a little piece from chapter 12 – probability.
Textbook practice homework for probability: from section 12.2 – 1, 3, 5, 13, 19; from section 12.4 – 25, 27

Example E. Suppose we measure the heights of 25 people and we define X = height in inches rounded to the nearest integer and get the following results:

height (in.)	64	65	66	67	68	69	70
frequency	5	7	6	4	1	0	2
probability							

The probability histogram would look like this:



We can calculate probabilities for various values of X.

a) The probability that X is not more than 66 is

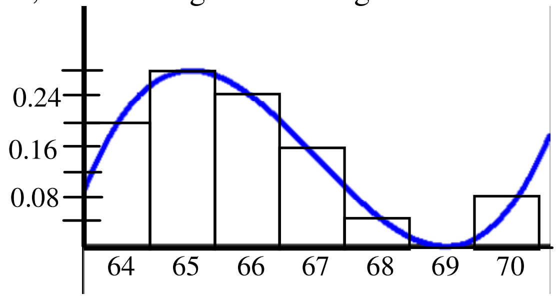
b) The probability that X is at least 68 is

The areas of the rectangles in the probability histogram represent probabilities. Note that the sum of the areas of all of the rectangles = 1.

{area of each bar} = {percentage of people having that height}
= {relative frequency of that height}
= {probability of a person picked at random having that height}.

In other words, we can equate probability with the idea of "area under the curve".

If we were to measure to the nearest half-inch, or tenth of an inch, or hundredth, or thousandth, etc., etc., etc., we'd get ever-more-narrow rectangles, and would get something more like a curve:



The area under the curve for a given interval would be the probability of people having heights within that interval. How can we find "area under the curve"? Integration!

For a continuous random variable X, with a ***probability density function f*(*x*)**, the probability that X will take on a value in an interval $a \le X \le b$ is $P(a \le X \le b) = \int_a^b f(x)\,dx$.

Example F: A continuous random variable has a probability density function $f(x) = 12x(1-x)^2$ on the interval $0 \le x \le 1$. 1) Show that $\int_0^1 12x(1-x)^2\,dx = 1$. 2) Find $P\left(0 \le X \le \frac{1}{2}\right)$. *answer*: $\frac{11}{16} = 0.6875$

Example G: Show that $f(x) = 2x - 2$ is a probability density function for $1 \le x \le 2$.

Example H. For a particular machine, its useful lifetime (random variable *T*) is modeled by probability density function $0.1e^{-0.1t}, \quad 0 \le t < \infty$.Find the probabilities that the machine will last 1) less than 2 years, 2) more than 2 but less than 4 years, and 3) more than 4 years.
answers: a) $-e^{-0.2}+1 \cong 0.181$, b) $-e^{-0.4}+e^{-0.2} \cong 0.148$, c) $e^{-0.4} \cong 0.670$

Side note: The three probabilities add to 1, as they should. (We must add the exact values, because the approximate answers have a rounding error.)

Exponential probability distributions have very nice characteristics that make them easy to work with. If we replaced the coefficient above 0.1 with *k*, we would get the general form of an exponential probability density function, $f(t)=ke^{-kt}$. In addition, for exponential probability, it is always true that the ***mean of random variable X*** (that is, the arithmetic **average** or **expected value**) is $\mathrm{E}(X)=\frac{1}{k}$.

Example H revisited. For a particular machine, its useful lifetime (random variable *T*) is modeled by probability density function $0.1e^{-0.1t}, \quad 0 \le t < \infty$.What is the expected lifetime of this machine? (Note: The phrases "expected lifetime", "average lifetime" and "mean lifetime" are equivalent.) *answer*: 10 years

Example I: The mean time (expected value) between hits at a website is 0.5 seconds. Find the exponential probability density function *f*(*x*) for random variable *X* = time between hits. (Note: Once we have the probability density function, we can use it to answer any questions about probability.) *answer*: $f(t)=2e^{-2t}$

Another often-useful probability density function is the normal density function, which graphs as the familiar bell-shaped curve. If we consider the special case where E(X) = $\mu = 0$ and standard deviation = $\sigma = 1$, we get what is called the standard normal distribution, $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, with its graph called the standard normal curve. There is no method for integrating to find area under the curve (and thus probabilities) of this standard normal probability density function, but a calculation using one of the more sophisticated approximation methods (covered in Math 121) has been done to construct tables of values.

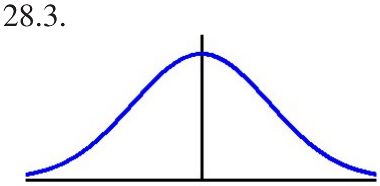
Rather than approximate values for every possible normal density function, the common practice is to convert everything to a standard normal distribution and use the same normal distribution table over and over. In the science of statistics, where things such as sampling distributions are normally distributed, data will be converted to the standard normal random variable Z, where $Z = \frac{X - \mu}{\sigma}$.

**Important note: For this Lecture, I won't be using the version of the standard normal probability table that is in the Appendix of your text. Instead, I'll be using the (much more common) version that is found at the end of this Lecture outline. This more common version is called a "cumulative probability distribution", and relies on the Fundamental Theorem of Calculus: $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$. Rather than find an antiderivative formula, we'll first look up the number values for $F(b)$ and $F(a)$ in the table, then use a calculator/computer function to do the same calculation for us.

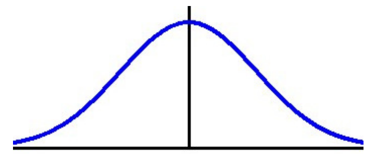
Examples J: (source http://www.cdc.gov/nchs/data/ad/ad347.pdf) "During the last 2 decades the prevalence of obesity and overweight has increased in the United States. This is in part due to a shift in the distribution of BMI (weight in kilograms divided by height in meters squared) of the entire population…Between the early1960s and 1999–2002 mean BMI for men 20–74 years of age increased from just over 25 to almost 28. Similarly, for women mean BMI increased from almost 25 to just over 28." Distribution is approximately normal. Using data from Table 10 of this report, in 1999-2002 mean BMI for adults 20 years and over = 27.95, with a standard deviation of 0.15.

a) Find the probability that an adult selected at random has a BMI between 27.7 and 28.3.

answer: 0.9426



b) Find the probability that an adult selected at random has a BMI greater than 28.3.
answer: 0.0099

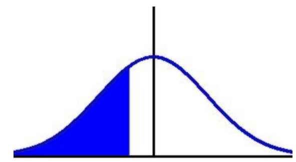


Textbook practice homework: from section 12.2 –19, 20 25a; from section 12.4 – 5, 7, 9, 25, 27

120 Lecture 6.5 Cumulative Standard Normal Distribution

Area Under a Standard Normal Curve to the Left of z $[P(Z < z)]$

$$P(a < Z < b) = F(b) - F(a).$$



z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

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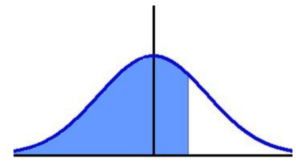

Area Under a Standard Normal Curve to the Left of z $[P(Z < z)]$

$$P(a < Z < b) = F(b) - F(a).$$

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998