Calculus 120, section 7.2 Partial Derivatives

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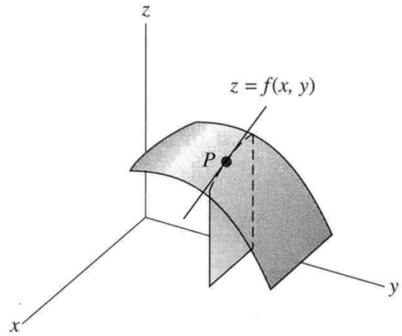
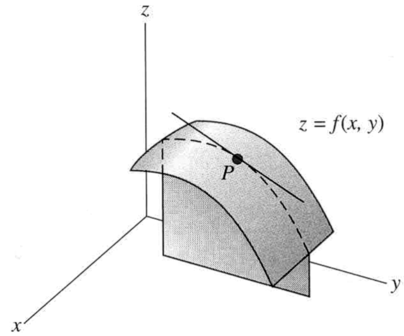
When working with functions of more than one variable, the question in calculus becomes: how can we evaluate the rate of change? The answer is called a *partial derivative*. Given a function $f(x, y, z)$, the partial derivative of f with respect to x, $\frac{\partial f}{\partial x}$, is found by treating all variables other than x as constants. The partial derivatives $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ have analogous definitions.

Example A: Given the function $f(x, y) = 2x + 2y$ find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. *Answers*: 2; 2

Example B: Given the function $f(x, y) = 9 - x^2 - y^2$ find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. *Answers*: $-2x$; $-2y$

A geometric interpretation of partial derivative is pictured below. In each figure $f(x, y)$ is the curved surface. In the figure on the left, with y treated as a constant, the tangent line goes the same general direction as the x-axis, and $\frac{\partial f}{\partial x}$ is the slope of that tangent at point P. With x treated as a constant, the tangent line goes the same general direction as the y-axis, and $\frac{\partial f}{\partial y}$ is the slope of that tangent at point P.





Example C: Given the function $f(x, y, z) = x^2yz - ze^{xy} + \frac{x}{y}\ln(z)$ find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

Answers: $2xyz - yze^{xy} + \frac{1}{y}\ln(z)$; $x^2z - xze^{xy} - \frac{x}{y^2}\ln(z)$; $x^2y - e^{xy} + \frac{x}{yz}$

Example D: For $f(x, y) = x^2 + 3xy + y + 7$, determine $\frac{\partial f}{dx}(5, 8)$ and $\frac{\partial f}{dy}(5, 8)$. *Answers*: 34; 16

Example E: The revenue generated by x model A speakers and y model B speakers is given by $R(x, y) = 100x + 150y - 0.03x^2 - 0.02y^2$ dollars. Determine the rate at which revenue will change with respect to the change in the number of model A speakers sold when 50 model A speakers and 40 model B speakers have been sold. *Answer*: \$97 per model A speaker

Example F (Example C from Lecture 7.1 revisited): For a particular manufacturing plant, the number of units produced is given by the Cobb-Douglas function $f(x, y) = 20x^{0.4}y^{0.6}$, where x = the number of units of labor and y = the number of units of capital. Find and interpret $\frac{\partial f}{\partial x}(1200, 2000)$ and $\frac{\partial f}{\partial y}(1200, 2000)$.

Answers: ≈11 units; ≈ 10 units

This partial derivative $\frac{\partial f}{\partial x}$ is called the *marginal productivity of labor*. At the current production level using 1200 units of labor and 2000 units of capital, if labor is increased by one unit, the level of production will increase by approximately 11 units.

This partial derivative $\frac{\partial f}{\partial y}$ is called the *marginal productivity of capital*. At the current production level using 1200 units of labor and 2000 units of capital, if capital is increased by one unit, the level of production will increase by approximately 10 units.

Example G: For $f(x, y) = xy^3 + ye^x$ determine $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$.

Just like regular derivatives, higher order partial derivatives can be found. The first two are called respectively the second partial derivative with respect to x and the second partial derivative with respect to y. The second two are often called "mixed partial derivatives". In this case, as with most functions, the two mixed partials are equal. (You should use this fact to check your answers.)

Answers: ye^x; $6xy$; $3y^2 + e^x$; $3y^2 + e^x$

Example H: For $g(x, y) = \ln(xy) + e^{xy}$ determine $\dfrac{\partial^2 f}{\partial y\, \partial x}$.

Just like regular derivatives, you need to know and be able to use the product rule, quotient rule and chain rule.
Answer: $xye^{xy} + e^{xy}$