Calculus 120, section 7.3 Maxima & Minima of Multivariable Functions

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A quick note to start: If you're at all unsure about the material from 7.1 and 7.2, *now* is the time to go back to review it, and get some help if necessary. You'll need *all of it* for the next three sections. Just like we had a first-derivative test and a second-derivative test to maxima and minima of functions of one variable, we'll use versions of first partial and second partial derivatives to determine maxima and minima of functions of more than one variable.

The first derivative test for functions of more than one variable looks very much like the first derivative test we have already used: If $f(x, y, z)$ has a relative maximum or minimum at a point (a, b, c) then all partial derivatives will equal 0 at that point. That is

$$\frac{\partial f}{\partial x}(a, b, c) = 0 \qquad \frac{\partial f}{\partial y}(a, b, c) = 0 \qquad \frac{\partial f}{\partial z}(a, b, c) = 0$$

Example A: Find the possible values of x, y, and z at which $f(x, y, z) = x^2 + 2y^3 + 3z^2 + 4x - 6y + 9$ has a relative maximum or minimum. *Answers*: (–2, –1, 0); (–2, 1, 0)

Example B: Find the possible values of x and y at which $f(x, y) = x^2 + 4xy + y^2 - 12y$ has a relative maximum or minimum. *Answer*: (4, –2); $z = 12$

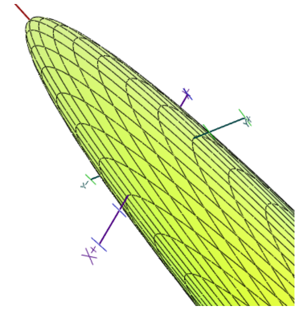
Notice that the example above asked for *possible* values. The first derivative test by itself is inconclusive. The second derivative test for functions of more than one variable is a good bit more complicated than the one used for functions of one variable. We'll apply it only to functions of two variables.

First calculate $D(x, y) = \frac{\partial^2 f}{\partial x^2} * \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$. Then, given a point (a, b) which is a possible extremum:

1. If $D(a, b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) > 0$ then $f(x, y)$ has a relative minimum at (a, b).
2. If $D(a, b) > 0$ and $\frac{\partial^2 f}{\partial x^2}(a, b) < 0$ then $f(x, y)$ has a relative maximum at (a, b).

3. If $D(a, b) < 0$ then $f(x, y)$ has neither a relative maximum or minimum at (a, b). (This is sometimes called a saddle point because of the shape.)
4. If $D(a, b) = 0$ then the test is inconclusive—we don't know what's happening at (a, b).
Remember that a second derivative is NOT a multiplication, but IS the derivative of a derivative.

Example C (Example H from 7.1 revisited): The 3-D graph of the function $f(x, y) = 9 - x^2 - y^2$ shows a relative maximum. We'll use the tests above to determine the location and verify that it is a maximum.



Example D: Use first and second derivative tests to determine points where $f(x, y) = 3x^3 + y^2 - 9x - 6y + 1$ has relative extrema. *Answers*: (–1, 3, –2) saddle point; (1, 3, –14) relative minimum.

Example E: Use first and second derivative tests to determine points where $f(x, y) = xy - x^2 - y^2 - x - 4y + 4$ has relative extrema. *Answer*: $(-2, -3, 11)$ relative maximum