Calculus 120, section 7.4 Lagrange Multipliers

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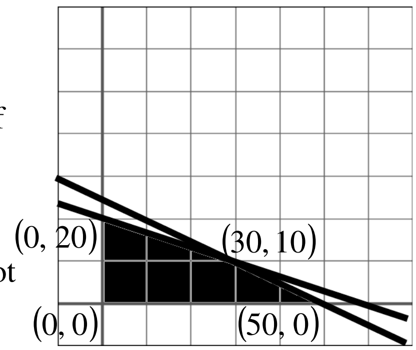
If you have taken Math 110 or 113 or a finite mathematics course you will have encountered linear programming. A typical word problem looks like this:

> A company manufactures two types of desks. Let x = the number of steel desks and let y = the number of wood desks. The profit for steel desks is \$80 each, and the profit for wood desks is \$175. The company wants to maximize its profit. Each steel desk requires 2 hours of assembly and 1 hour of finishing. Wood desks require 4 hours of assembly and 3 hours of finishing. The company has 100 work-hours available for assembly and 60 work-hours available for finishing.

The company's goal (i.e. objective) is to maximize profit: $P = 80x + 175y$. In theory, this function has no maximum: make more desks = make more money. In the real world, however, there are limitations (i.e. constraints): the number of employees and therefore the number of desks that can be made has an upper limit. The hours available for assembly is expressed in the assembly constraint $2x + 4y \le 100$. The hours available for finishing is expressed in the finishing constraint $x + 3y \le 60$. In addition, the number of each type of desk made cannot be negative: $x \ge 0$ and $y \ge 0$. The "system of constraints" which illustrates the "feasible set" is graphed to the right, with corners labeled.



If we were to graph the *level curves* for the profit function $P(x, y) = 80x + 175y$, they would appear as a series of parallel lines: all with the same slope but representing varying levels of profit. The maximum possible (i.e. feasible) profit is represented by the level curve where $P = 4150$ that passes through the corner (30, 10).

You won't be asked to do a linear programming question in this class, but you will need some of the same algebra skills, such as solving a system of equations.

Rather, in calculus, while we're still looking for some optimum value (maximum or minimum), neither the objective nor the constraints are likely to be linear functions, and we'll need somewhat more involved methods of finding the maximum or minimum of the objective, within the given constraints. Given an objective function f and a constraint function g the process looks like this:

- Identify the objective function—it's the one that needs to be maximized or minimized.
- Write the constraint function in the form $g = 0$.
- Create a function F = objective + λ(constraint) where λ is the Lagrange multiplier.
- Find all first partial derivatives, including with respect to λ, and set them equal to 0.
- Solve the resulting system of equations for all variables, including λ.

It will usually be best to solve the first equations for λ and set them equal to each other, using a series of substitutions to find the rest of the values.

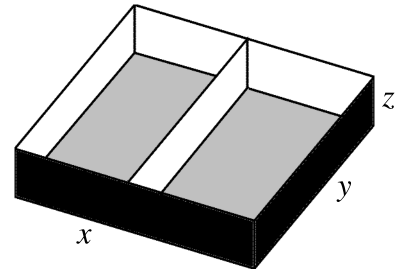
Example A: Find the minimum value of $f(x, y) = 2x^2 + y^2 + 7$ subject to the constraint $g(x. y) = x + y + 18$.
Answer: $f(-6, -12) = 223$

Example B: For a specified product, the Cobb-Douglas production function is $f(x, y) = 600x^{2/3}y^{1/3}$ where x = the number of units of labor and y = the number of units of capital. The cost of labor is \$400 per unit and the cost of capital is \$200 per unit. The company wants to make 54000 of their product at the lowest possible cost. Determine the number of units of labor and the number of units of capital that would minimize cost.
Don't be fooled into thinking that the letter *f* always indicates the objective function! The letters used are chosen arbitrarily. *Answer*: $C(90, 90)$ = \$54000

Note that at $(x, y) = (90, 90)$, $\frac{\partial C}{\partial x}$ = marginal cost of labor = 400, and $\frac{\partial C}{\partial y}$ = marginal cost of capital = 400.

Also, when $(x, y) = (90, 90)$, $\lambda = \frac{90^{1/3}}{90^{1/3}} = 1$. In this case λ represents the marginal cost of productivity: If one more unit is produced, the cost per unit is \$1.

Example C: We want to make a rectangular open box with one partition in the middle, as illustrated in the picture, from 162 in^2 of cardboard. Find the dimensions that would maximize the volume. (Okay, this isn't much like a real-world problem. If you'd rather, you can change it to "cargo container and sheet metal".)



Answer: 9 in by 6 in by 3 in

Note that $\frac{\partial V}{\partial x} = yz \Rightarrow$ Increasing length by 1 increases volume by 18 in^3. $\frac{\partial V}{\partial y} = xz \Rightarrow$ Increasing width by 1 increases volume by 27 in^3. $\frac{\partial V}{\partial z} = xy \Rightarrow$ Increasing height by 1 increases volume by 54 in^3. (All easily verifiable). λ = marginal volume with respect to surface area = $\lambda = \frac{-xz}{x+3z} = \frac{-9*3}{9+3*3} = -\frac{27}{18} = -\frac{3}{2}$: At the optimal level, changes in the dimensions within the given constraint would decrease the volume.