Calculus 140, section 2.5 Continuity, IVT and Bisection

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Definition 2.8: "A function f is **continuous** at a number a in its domain if $\lim_{x\to a} f(x) = f(a)$.

A function f is **discontinuous** at a number a in its domain if f is not continuous."

Note that there are two implicit conditions necessary for continuity:

1) f is defined at a 2) $\lim_{x\to a} f(x)$ exists

Hopefully, it is obvious that that all of the functions we are used to dealing with (polynomial, rational, basic trigonometric, natural exponential, natural logarithm) are continuous at every point *in their domains*.

The definition of continuity, combined with the properties of limits (section 2.3, Theorem 2.2), lead us to a statement about continuity.

Theorem 2.9: "Suppose f and g are continuous at a, and let c be any number. Then $f + g$, cf, and fg are continuous at a. If $g(a) \neq 0$, then f/g is continuous at a."

From the Substitution Rule of section 2.3, we have $\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right) = g(f(a))$.

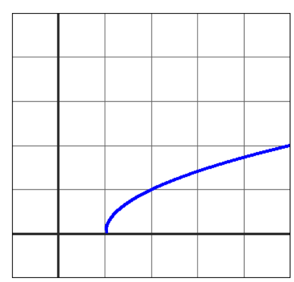
Theorem 2.10: "If f is continuous at a, and g is continuous at $f(a)$, then $g \circ f$ is continuous at a."

As a result, we can say that esoteric-looking functions such as $\sin(x^2)$, $\sin^2 x$, and $e^{\sin x}$ are continuous at all real numbers.

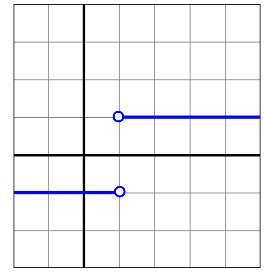
What if a function does not have a domain of all real numbers? How can we address the notions of continuity and discontinuity?

Definition 2.11: "A function f is **continuous from the right** at a point a in its domain if $\lim_{x\to a^+} f(x) = f(a)$. A function f is **continuous from the left** at a point a in its domain if $\lim_{x\to a^-} f(x) = f(a)$."

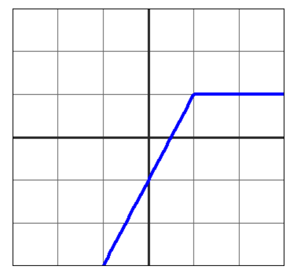
2.4 Example A revisited: Determine the continuity of $f(x) = \sqrt{x-1}$ at $x = 1$.



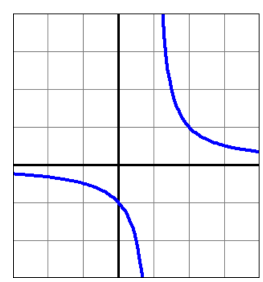
2.4 Example B revisited: Determine the continuity of $f(x)=\dfrac{|x-1|}{x-1}$ at $x = 1$.



2.4 Example B extended revised and revisited: Determine the continuity of $f(x)=\begin{cases} 2x-1 & \text{for } x<1 \\ 1 & \text{for } x \geq 1 \end{cases}$ at $x = 1$.

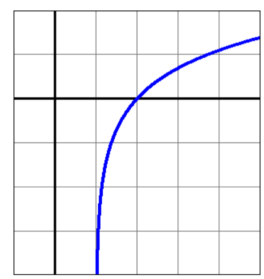


2.4 Example C revisited: Determine the continuity of $f(x)=\dfrac{1}{x-1}$ at $x = 1$.



2.4 Example C extended revisited: Determine the continuity of $f(x) = \dfrac{x-1}{x^2-1}$ at $x = 1$.

2.4 Example D revisited: Determine the continuity of $f(x) = \ln(x-1)$ at $x = 1$.



Definition 2.12: "A function is **continuous on an open interval (*a*, *b*)**, or simply **continuous on (*a*, *b*)**, if it is continuous at every point in (a, b).
A function is **continuous on a closed interval [*a*, *b*]**, or simply **continuous on [*a*, *b*]**, if it is continuous at every point in (a, b) and is also continuous from the right at a and continuous from the left at b."

Examples A, B, B extended, C, C extended and D one more time: Go back to each of these and specify open and closed intervals on which each is continuous.

Text Example 2: Show that $f(x) = \sqrt{4-x^2}$ is continuous on $[-2, 2]$.

(Look this one over as part of your "reading of each section we cover".)

We now come to the *big* piece of section 2.5: the **Intermediate Value Theorem**.
Theorem 2.13: "Suppose f is continuous on a closed interval $[a, b]$. Let p be any number between $f(a)$ and $f(b)$, so that $f(a) \le p \le f(b)$ or $f(b) \le p \le f(a)$. Then there exists a number c in $[a, b]$ such that $f(c) = p$."

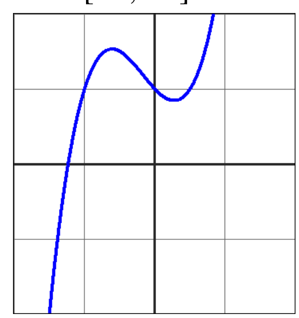
The text refers to the graph of a function crossing a horizontal line, and a person crossing an infinitely long river. Another way to put it: If f is continuous on $[a, b]$, then to get from $f(a)$ to $f(b)$ one must cross every function value in between.

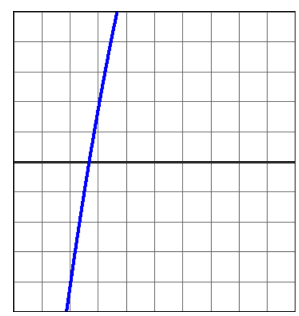
In Math 115 Precalculus, the IVT (even though left unnamed) was the basis for using a table of signs to solve inequalities, and to determine whether polynomial and rational functions lay above or below the x-axis. (See text Example 3 and Example 4.)

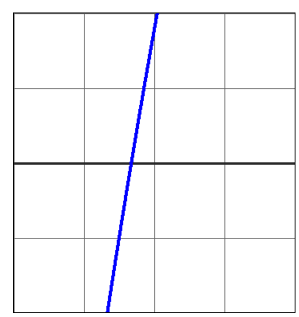
Eventually, we'll be interested in using the concept that, if the slope of a tangent to a function f is negative in one place, and positive in another, and as long as f is continuous between the two, then the slope of a tangent of f must equal 0 somewhere in between also. That is, the graph of f must have either a relative maximum or a relative minimum.

For the moment, we'll be content with using the IVT as the foundation of a process for finding a zero (or root) of a generic function: the **Bisection Method**. Given an interval $[a, b]$ on which f is continuous, and the knowledge that one of $f(a)$ and $f(b)$ is positive while the other is negative, we'll narrow our search for the value c such that $f(c) = 0$ by cutting the interval $[a, b]$ into half, then half again, then half again, until we're as close as we want to be. (Consult the text for definitions of algorithm, tolerance, and explanation of the steps of the bisection method.)

Example E: The function $f(x) = 2x^3 + x^2 - x + 1$ has exactly one zero, located in the interval $[-2, -1]$. Use the bisection method to estimate it within a tolerance of $\varepsilon = 0.1$.







Interval	**Length**	**Midpoint *c***	**$f(c) \approx$?**

The text notes, at the end of the section, that zeroes of functions can be estimated on a graphing calculator by using the Zoom and Trace functions.